REPORT 1113

SPECTRUM OF TURBULENCE IN A CONTRACTING STREAM[1]

By H. S. Ribner and M. Tucker

SUMMARY

The spectrum concept is employed to study the selective effect of a stream contraction on the longitudinal and lateral turbulent velocity fluctuations of the stream. By a consideration of the effect of the stream contraction on a single plane sinusoidal disturbance wave, mathematically not dissimilar to a triply periodic disturbance treated by G. I. Taylor, the effect on the spectrum tensor of the turbulence and hence on the correlation tensor is determined. Lack of interference between waves follows from the postulation of a low level of turbulence; this and the assumption of an inviscid fluid imply neglect of decay effects. The compressibility of the main stream is taken into account, but the density fluctuations associated with the turbulence are assumed to be negligible; this would be the case if the turbulence originated from wakes and boundary layers in the very low-speed portion of the flow. For an axisymmetric contraction and a particular isotropic initial turbulence, some explicit results are obtained. The one-dimensional longitudinal spectrum is found to be distorted (as well as reduced in amplitude) with its peak shifted well to the right of the initial position above the zero of the wave-number scale. The selective effect of the contraction on the mean square longitudinal and lateral components of turbulent velocity is found to be given uniquely when the initial turbulence is isotropic, regardless of the details of the spectrum. If the initial spectrum is anisotropic, as, for instance, that produced by a damping screen, then the selective effect is altered.

In a crude extension, decay effects outside the scope of the theory are allowed for in first approximation. With this extension, a comparison with experiment is made of the selective effect on turbulent intensity where the estimated decay effects are comparable with the contraction effects. The agreement is good for the longitudinal component, very poor for the lateral component, the experimental data themselves being in conflict.

INTRODUCTION

The generation of a wind-tunnel flow is always accompanied by a certain amount of turbulence; this is one respect in which the flow fails to simulate free-flight conditions. Measurements in the tunnel, particularly those sensitive to boundary-layer behavior, are known to be affected by this turbulence. Accordingly, the tunnel designer attempts to reduce the intensity to the lowest practicable level. The use of honeycombs and damping screens in a large low-speed section (settling chamber) followed by a sharp contraction to the much-higher-speed working section is known to be effective. The honeycombs and screens located in a low-speed section reduce the absolute level of the turbulence with little drag penalty; then the relative level is greatly reduced by the large gain in tunnel speed through the contraction, aside from any effect of the contraction on the absolute level.

Once the characteristics of honeycombs and screens are known, the further quantitative estimate of the reduction in turbulence involves a knowledge of the effect of the tunnel contraction[2] on the turbulence. It is known that the longitudinal component of the turbulence is greatly reduced (in absolute value) by the contraction; the behavior of the lateral component appears, on the other hand, to vary from no change to a substantial increase. Prandtl (ref. 1) obtained a quantitative estimate of the first effect by considering the conservation of energy for a perturbed longitudinal filament: if the initial stream speed is U, the filament speed $U+u$, with $u \ll U$, and the final stream speed is lU, then the final filament speed must be $lU+l^{-1}u$; that is, the contraction reduces the longitudinal perturbation velocity u by the factor l^{-1}. For the lateral effect, Prandtl applied conservation of momentum to a small rotating cylinder of the fluid, with its axis cross stream, as the fluid traversed the tunnel contraction. He concluded that the lateral perturbation velocity v is increased by a factor $\sqrt{l}$.

Prandtl's considerations on the effect of a stream contraction were limited, as has been noted, to particular idealized "eddies." G. I. Taylor (ref. 2) later attempted more realism by treating a mathematically defined model of turbulence which amounted to vortices in parallelepiped partitions arranged in a regular three-dimensional array. The changes in vorticity on traversing the contraction were determined from a theorem based on conservation of circulation for an inviscid fluid; the corresponding altered turbulent velocity pattern was then calculated. The final result of the analysis consisted in expressions for the root-mean-square longitudinal and lateral turbulent velocity components u' and v' downstream of the contraction expressed as ratios of the corresponding values upstream.

[1] Supersedes NACA TN 2606, "Spectrum of Turbulence in a Contracting Stream," by H. S. Ribner and M. Tucker, 1952.

[2] The considerations of this paper are not limited to a wind-tunnel contraction: they may be applied to any stream tube of varying cross section large compared with the scale of the turbulence.

The initial condition of isotropic turbulence (mean values unaffected by rotation or reflection of axes) was approximated by specifying the vortex partitions to be cubical. For this case the reduction in the longitudinal component u' was found to vary more nearly like 1.5 l^{-1} than the value l^{-1} suggested by Prandtl. No explicit result was found for the variation of the lateral component, however: the calculations contained a free parameter.

Taylor's results for the longitudinal component agreed fairly well with the experimental data then available, but it is now considered that the measurements were made too close behind the screens for the screen-produced turbulence to have been isotropic. On theoretical grounds, the objection to Taylor's theory is threefold: first, the decay processes of turbulent mixing and viscous dissipation, which result in a reduction of the mean intensity with axial distance in the wind tunnel, are neglected; second, the assumed model of turbulence fails to exhibit the spatial and temporal randomness of actual turbulence; third, no choice of the parameters in Taylor's model corresponds to isotropy. In a sense all three objections apply likewise to Prandtl's results: no model was employed in his considerations, and hence no distinctions between the effects of isotropy and anisotropy were made.

The second and third objections can be removed by working, not with a model of turbulence, but instead with a Fourier integral representation of a random turbulent field. The integral can be interpreted as a superposition of plane transverse sinusoidal waves of all wave lengths and with apparently random orientations and phases. This aggregate of plane waves constitutes the (three-dimensional) spectrum of the turbulence. Only the statistical aspects of this spectrum will be known, not, for example, the detailed phase relations. Mean-square velocity components may be obtained by an integral of certain spectrum functions in which the phase relations are suppressed; these functions are included in the 'spectrum tensor' (ref. 3).

Taylor's concepts may be applied to find the effect of a stream contraction on a single plane wave. The effect under the assumptions is linear; therefore, the superposition implied by the Fourier integral may be employed to obtain the contraction effect on a field of turbulence. In particular, if the initial spectrum tensor is known, the final spectrum tensor is determined; the initial and final mean-square velocity components then result from quadratures. Indeed, from the same information, changes due to the stream contraction in correlations of velocity at different points may be calculated: use is made of the fact that the correlation tensor is an inverse Fourier transform of the spectrum tensor (ref. 3).

Accordingly, the injection of the spectrum point of view into Taylor's original concept of the contraction effect makes possible a more realistic calculation of the changes in mean-square velocity components. In addition, it provides much more detailed information concerning changes in the statistical properties of the turbulence; that is, it provides the changes in the spectrum tensor and in the correlation tensor.

The ideas just outlined are developed in the present paper. The first section is devoted to an account of turbulent spectrum analysis in a form specially adapted to the analysis of the contraction effect. In this account, which is a generalization of a development in reference 4, the role of the spectrum tensor is subordinated to that of the individual Fourier components (plane waves) in contradistinction to the customary treatment. This approach has perhaps an auxiliary merit in providing some better physical insight into the significance of the spectrum tensor.

Next the effect of a stream contraction on a single plane wave is calculated by an application of Taylor's concepts. The treatment is slightly more general in that compressibility of the main stream is allowed for. The density fluctuations associated with the turbulence are assumed to be negligible; this would be the case if the turbulence originated entirely from boundary layers and wakes in the very-low-speed portion of the flow. Following Taylor, the problem is linearized by postulating a sufficiently weak turbulence so that the self-distortion of the turbulent eddies is small compared with the distortion imposed by the contraction of the main stream; this together with the assumption of an inviscid fluid implies neglect of the decay of the turbulence.

In succeeding sections the spectrum and correlation tensors downstream of the contraction are expressed in terms of the corresponding initial tensors. For the special case of an axisymmetric contraction and isotropic initial turbulence, the ratios of the root-mean-square longitudinal and lateral velocity fluctuations downstream and upstream are obtained explicitly in terms of the parameters defining the contraction. For a particular subcase where the initial isotropic spectrum tensor is specified, the corresponding 'one-dimensional' spectrums (as would be recorded by stationary hot-wire probes) upstream and downstream of the contraction are calculated; the specification is such that the upstream one-dimensional spectrum corresponds to experiment (ref. 5, p. 35) in a number of cases of isotropic turbulence.

Most of the calculated contraction effects are amenable to experimental checks either directly or indirectly. The available experimental data, however, are limited to the changes in the root-mean-square velocities. A comparison with these experimental data is given with an estimated allowance for decay effects outside the scope of the theory. Design curves of the changes in the root-mean-square velocity components neglecting decay are included for engineering purposes.

SPECTRUM ANALYSIS

Representation of turbulence by superposition of plane sinusoidal waves.—Suppose q_1, q_2, q_3 represent the components of velocity in a turbulent field; that is q_1, q_2, and q_3 vary in an apparently random manner in space and time, and the mean values $\overline{q}_1=\overline{q}_2=\overline{q}_3=0$. Subject to certain conditions, a snapshot of this field at any instant can be

represented as a set of three-dimensional Fourier integrals

$$q_\alpha(x_1, x_2, x_3)=\int\int_{-\infty}^{\infty}\int Q_\alpha(k_1, k_2, k_3)\,e^{i(k_1x_1+k_2x_2+k_3x_3)}\,dk_1dk_2dk_3$$

where $\alpha=1$, 2, or 3 and the significance of k_1, k_2, and k_3 will be brought out later. A continuous representation of the turbulent field is obtained by allowing the Q_α to vary with time.

It will be convenient to abbreviate the Fourier integral to

$$q_\alpha(\underline{x})=\int\int_{-\infty}^{\infty}\int Q_\alpha(\underline{k})\,e^{i\underline{k}\cdot\underline{x}}d\tau(\underline{k}) \tag{1a}$$

and to introduce the companion equation

$$Q_\alpha(\underline{k})=8\pi^3\int\int_{-\infty}^{\infty}\int q_\alpha(\underline{x})\,e^{-i\underline{k}\cdot\underline{x}}d\tau(\underline{x}) \tag{1b}$$

where $\underline{k}=k_1,\ k_2,\ k_3$; $\underline{x}=x_1,\ x_2,\ x_3$; $d\tau(\underline{x})=dx_1dx_2dx_3$. The second equation allows, in principle at least, the coefficients $Q_\alpha(\underline{k})$ to be calculated. Mathematically, $q_\alpha(\underline{x})$ and $Q_\alpha(\underline{k})$ are termed three-dimensional Fourier transforms of each other; use will be made of this relation later.

The velocity components q_α are connected by the condition of continuity. In many cases of practical interest, these turbulent velocities originate from boundary layers and the wakes of obstacles in flows of low subsonic speed, so that associated density fluctuations may be ignored; this is still permissible when the turbulence so produced is transported by a high-speed stream. Thus the incompressible form of the continuity equation may be used and the result is

$$Q_1k_1+Q_2k_2+Q_3k_3=0$$

This relation may be written more compactly as

$$\sum_\alpha Q_\alpha k_\alpha=0 \tag{2}$$

Physical interpretation.—The amplitude components Q_α are complex in general. According to equation (1b), then, the requirement that the velocity components q_α be real implies that $Q_\alpha(-\underline{k})$ is the complex conjugate of $Q_\alpha(\underline{k})$. If corresponding terms for $\underline{k}$ and $-\underline{k}$ in equation (1a) are paired, their sum is thus equal to the real quantity

$$2(Re\ Q_\alpha)\cos(\underline{k}\cdot\underline{x})-2(Im\ Q_\alpha)\sin(\underline{k}\cdot\underline{x}) \tag{3}$$

The imaginary parts cancel in the pairing, which implies that they contribute nothing to the integral. Expression (3) represents a pair of plane standing waves, a cosine wave and a sine wave, with normals in the direction $\underline{k}=(k_1,\ k_2,\ k_3)$, where $\underline{x}=(x_1,\ x_2,\ x_3)$ is the radius vector to any point. The vector $\underline{k}$ is termed the wave-number vector and its magnitude k simply the wave number; the corresponding wave length is 2π divided by the wave number. Since $\underline{k}$ is perpendicular to the wave front, it is sometimes referred to herein as the 'wave normal.'

The continuity condition, equation (2), states that both the real part ($Re\ Q_\alpha$) and the imaginary part ($Im\ Q_\alpha$) of the amplitude vector $\underline{Q}=Q_\alpha=(Q_1,\ Q_2,\ Q_3)$ are perpendicular to the wave normal $\underline{k}$; that is, both waves of expression (3) are transverse. For each wave any one of the parallel planes containing both the local velocity vector $\underline{q}$ and the wave-number vector $\underline{k}$ (which is perpendicular to $\underline{Q}$ and hence to $\underline{q}$) is called the plane of polarization. The cosine wave (real part) and sine wave (imaginary part) may be polarized in different planes in general; the necessary and sufficient condition that they be polarized in the same plane is

$$\frac{Re\ Q_1}{Im\ Q_1}=\frac{Re\ Q_2}{Im\ Q_2}=\frac{Re\ Q_3}{Im\ Q_3} \tag{4}$$

Equations (1) are now seen to represent a superposition of plane sinusoidal waves (Fourier components) with all orientations of the wave fronts (all directions of the wave normal $\underline{k}$) and all wave lengths (all wave numbers k). Each wave is transverse, and all planes of polarization are permitted. For each value of $\underline{k}$ there exist a cosine wave and a sine wave; their respective amplitudes and planes of polarization are different in general. The complex amplitude components $Q_\alpha(\underline{k})$ express, in their real and imaginary parts, how the respective amplitudes and planes of polarization vary with the wave-front orientation and the wave number.

Mean values: correlation tensor.—Consider the spatial[3] mean value of the product of the velocity component q_α at $\underline{x}$ and the velocity component q_β' at $\underline{x}'=\underline{x}+\underline{r}$ as $\underline{x}$ varies but the separation $\underline{r}$ of the two points remains fixed during the averaging process; this mean value is called a velocity correlation and is given the symbol $R_{\alpha\beta}(\underline{r})$. There are nine such correlations, corresponding to $\alpha=1$, 2, 3, and $\beta=1$, 2, 3. The form $R_{\alpha\beta}(\underline{r})$ has been shown to transform like a second-order tensor and has been designated (sometimes divided by $\overline{q^2}$) as the 'correlation tensor' (ref. 6).

Evaluation of correlation tensor in terms of spectral quantities.—The basis of the idea that the correlation tensor might be expressed in terms of individual-wave parameters is drawn from reference 4. In that paper mean-square values of q_1, q_2, q_3 in terms of such parameters are discussed; these constitute the diagonal terms of $R_{\alpha\beta}(\underline{r})$ evaluated at $\underline{r}=0$. The following derivation of $R_{\alpha\beta}(\underline{r})$ amounts to a generalization of the derivation of $R_{11}(0)$ given in that reference.

Assume that the turbulence is confined to a large, but not infinite, parallelepiped of edges $2D_1$, $2D_2$, $2D_3$ and vanishes everywhere outside.[4] The space average $R_{\alpha\beta}(\underline{r})$ is to be

[3] If the statistical properties of the turbulence are independent of position (homogeneous turbulence) and time-independent, an average at a given time over all space equals an average at a given point (or pair of correlated points) over all time; a proof is given in appendix B. If the statistical properties vary slowly with time, the space average will still approximate a time average over an interval just long enough to smooth out the fluctuations.

[4] The space integral of the square of any component velocity is thereby bounded, which is a prerequisite for the existence of a Fourier integral, equations (1); in other words, equation (1b) shows that Q_α must depend on the volume $8D_1D_2D_3$ of the region within which q_α differs from zero: if this volume is infinite Q_α is infinite, and the Fourier integral, equation (1a), does not exist.

taken for all points $\underline{x}$ in the interior, the separation $\underline{r}$ remaining constant and very small compared with $D_{1,2,3}$. The average is

$$R_{\alpha\beta}(\underline{r})\equiv\overline{q_\alpha(\underline{x})q_\beta(\underline{x}+\underline{r})}=\frac{1}{8D_1D_2D_3}\int_{-D_3}^{D_3}\int_{-D_2}^{D_2}\int_{-D_1}^{D_1}\left\{\iiint_{-\infty}^{\infty}\iiint_{-\infty}^{\infty}Q_\alpha(\underline{k})Q_\beta(\underline{k}')e^{i[\underline{k}\cdot\underline{x}+\underline{k}'\cdot(\underline{x}+\underline{r})]}d\tau(\underline{k})d\tau(\underline{k}')\right\}dx_1dx_2dx_3$$

or, interchanging the order of integration,

$$=\frac{1}{D_1D_2D_3}\iiint_{-\infty}^{\infty}\iiint_{-\infty}^{\infty}Q_\alpha(\underline{k})Q_\beta(\underline{k}')e^{i\underline{k}'\cdot\underline{r}}\,\frac{\sin(k_1+k_1')D_1}{k_1+k_1'}\cdot\frac{\sin(k_2+k_2')D_2}{k_2+k_2'}\cdot\frac{\sin(k_3+k_3')D_3}{k_3+k_3'}\,d\tau(\underline{k})d\tau(\underline{k}')$$

Put $K_1=k_1+k_1'$, $K_2=k_2+k_2'$, and $K_3=k_3+k_3'$. Then recalling that $\underline{k}=(k_1,\ k_2,\ k_3)$, $d\tau(\underline{k})=dk_1dk_2dk_3$, etc., and $\underline{k}'\cdot\underline{r}=k_1'r_1+k_2'r_2+k_3'r_3$

$$R_{\alpha\beta}(r)=\frac{1}{D_1D_2D_3}\iiint_{-\infty}^{\infty}Q_\alpha(k_1,k_2,k_3)\left\{\iiint_{-\infty}^{\infty}Q_\beta(K_1-k_1,K_2-k_2,K_3-k_3)\right.$$
$$\left.e^{i[(K_1-k_1)r_1+(K_2-k_2)r_2+(K_3-k_3)r_3]}\,\frac{\sin K_1D_1}{K_1}\,\frac{\sin K_2D_2}{K_2}\,\frac{\sin K_3D_3}{K_3}\,dK_1dK_2dK_3\right\}dk_1dk_2dk_3$$

The inner integral approaches a limit as the parallelepiped edges $2D_1$, $2D_2$, $2D_3$ become indefinitely large. For this limiting case the correlation tensor is then

$$R_{\alpha\beta}(\underline{r})=\lim_{D\to\infty}\iiint_{-\infty}^{\infty}\frac{\pi^3}{D_1D_2D_3}\,Q_\alpha(k_1,k_2,k_3)Q_\beta(-k_1,-k_2,-k_3)e^{-i\underline{k}\cdot\underline{r}}dk_1dk_2dk_3=\lim_{\tau\to\infty}\iiint_{-\infty}^{\infty}\frac{8\pi^3}{\tau}\,Q_\alpha(\underline{k})Q_\beta^*(\underline{k})e^{-i\underline{k}\cdot\underline{r}}d\tau(\underline{k}) \qquad (5)$$

where $Q_\beta^*(\underline{k})[=Q_\beta(-\underline{k})]$ is the complex conjugate of $Q_\beta(\underline{k})$ and τ is the volume $8D_1D_2D_3$ of the parallelepiped.

Identification of $\lim\limits_{\tau\to\infty}\dfrac{8\pi^3}{\tau}\,Q_\alpha Q_\beta^*$ with spectrum tensor.—If the field of turbulence is homogeneous, the correlations should be independent of the volume τ averaged over when this volume is sufficiently large; thus $\lim\limits_{\tau\to\infty}\dfrac{8\pi^3}{\tau}\,Q_\alpha Q_\beta^*$ should exist. From the mathematical standpoint, equation (5) shows that $R_{\alpha\beta}(\underline{r})$ is the Fourier transform of $\lim\limits_{\tau\to\infty}\dfrac{8\pi^3}{\tau}\,Q_\alpha(\underline{k})Q_\beta^*(\underline{k})$, and conversely; since it is known that the integral of $|R_{\alpha\beta}(\underline{r})|$ over all $\underline{r}$ is finite, the inverse Fourier transform relation ensures that $\lim\limits_{\tau\to\infty}\dfrac{8\pi^3}{\tau}\,Q_\alpha Q_\beta^*$ exists. Finally, since the Fourier transform of a function is unique, $\lim\limits_{\tau\to\infty}\dfrac{8\pi^3}{\tau}\,Q_\alpha(\underline{k})Q_\beta^*(\underline{k})$ may be identified with the form $\Gamma_{\alpha\beta}(\underline{k})$ defined by Batchelor (ref. 3) as the Fourier transform of $R_{\alpha\beta}(\underline{r})$. The form $\Gamma_{\alpha\beta}(\underline{k})$ is known as the spectrum tensor. The Fourier transform relations connecting the spectrum tensor and the correlation tensor are summarized as

$$\Gamma_{\alpha\beta}(\underline{k})=\frac{1}{8\pi^3}\iiint_{-\infty}^{\infty}R_{\alpha\beta}(\underline{r})e^{i\underline{k}\cdot\underline{r}}d\tau(\underline{r}) \qquad (6a)$$

$$R_{\alpha\beta}(\underline{r})=\iiint_{-\infty}^{\infty}\Gamma_{\alpha\beta}(\underline{k})e^{-i\underline{k}\cdot\underline{r}}d\tau(\underline{k}) \qquad (6b)$$

By use of the Fourier transform relation, Batchelor demonstrated that $\Gamma_{\alpha\beta}$ is a second-order tensor and obtained a number of its properties. Thus, for example, $\Gamma_{\alpha\beta}$ is complex, in general, with $\Gamma_{\beta\alpha}=\Gamma_{\alpha\beta}^*$, and the diagonal elements $\Gamma_{\alpha\alpha}$ are real; also, $\Gamma_{\alpha\beta}(-\underline{k})=\Gamma_{\beta\alpha}(\underline{k})$. It is of interest to observe that these same properties result immediately from the identification of $\lim\limits_{\tau\to\infty}\dfrac{8\pi^3}{\tau}\,Q_\alpha Q_\beta^*$ with $\Gamma_{\alpha\beta}$. Thus $Q_\alpha Q_\beta^*$ is complex, in general; $Q_\beta Q_\alpha^*$ equals $[Q_\alpha Q_\beta^*]^*$; and $Q_\alpha Q_\alpha^*$ is, of course, real. Furthermore, since $Q_\alpha(-\underline{k})=Q_\alpha^*(\underline{k})$, $Q_\alpha(-\underline{k})Q_\beta^*(-\underline{k})$ equals $Q_\alpha^*(\underline{k})Q_\beta(\underline{k})$; hence $\Gamma_{\alpha\beta}(-\underline{k})=\Gamma_{\beta\alpha}(\underline{k})$.

The distinction between cases where $\Gamma_{\alpha\beta}$ is real and cases where it is complex may be given a physical interpretation. The product $Q_\alpha Q_\beta^*$, and correspondingly $\Gamma_{\alpha\beta}$, is seen to be real when the condition equation (4) is satisfied. This implies that the cosine wave and sine wave associated with wave number $\underline{k}$ are polarized in the same plane. The alternative condition where $Q_\alpha Q_\beta^*$, and hence $\Gamma_{\alpha\beta}$, is complex implies polarization of corresponding cosine and sine waves in different planes. The velocity pattern of such a pair of waves is quite interesting: successive velocity vectors along a line in the direction of the wave normal $\underline{k}$ turn progressively about this line in spiral fashion; the tips of the vectors trace out a helical curve on a cylinder of oval cross section.

Energy spectral density.—Each of the diagonal elements Γ_{11}, Γ_{22}, and Γ_{33} of the spectrum tensor $\Gamma_{\alpha\beta}$ may be interpreted as an energy spectral density. Thus, according to equation (6b)

$$R_{11}(0)\equiv\overline{u_1^2}=\iiint_{-\infty}^{\infty}\Gamma_{11}(\underline{k})d\tau(\underline{k})$$

Therefore, the differential $\frac{1}{2}\,\Gamma_{11}d\tau(\underline{k})$ represents the contribution to the kinetic energy component $\frac{1}{2}\,\overline{u_1^2}$ per unit mass made by waves with wave number within the range $d\tau(\underline{k})$.

One-dimensional spectrum.—The elements of the three-dimensional spectrum tensor are not directly measurable; they may be obtained by taking the Fourier transform of the measured correlation tensor. A hot-wire probe placed in the moving stream will, however, develop a fluctuating output voltage whose (one-dimensional) frequency spectrum (ref. 7) is related to a diagonal element of the three-dimensional spectrum tensor. Thus by equation (6b) the contribution to the mean-square velocity component $\overline{q_\alpha^2}(=R_{\alpha\alpha}(0))$ from all waves with wave-number components in the direction of the x_1-axis between $|k_1|$ and $|k_1|+|dk_1|$ is

$$F_\alpha(k_1)dk_1 \equiv 2\left(\int\int_{-\infty}^{\infty}\Gamma_{\alpha\alpha}dk_2dk_3\right)dk_1 \tag{7}$$

the factor of 2 accounting for suppression of negative values of k_1. The function $F_\alpha(k_1)$ is the one-dimensional spectrum corresponding to the velocity component q_α; the values $\alpha=1, 2, 3$ correspond respectively to the longitudinal and two lateral spectrums. The particular spectrum obtained depends on the arrangement of the hot-wire probe elements.

EFFECT OF STREAM CONTRACTION

Consider now that the turbulent velocity pattern q_1, q_2, q is carried along by an inviscid general stream with velocity $U(x_1)$ in the x_1-direction. Consider also that q_1, q_2, q_3 are so small that their effect on the streamlines may be neglected as the flow traverses a wind-tunnel contraction. The contraction will, however, distort the shape of fluid elements. (See fig. 1.) The vorticity distribution will be forced to alter accordingly to conserve the circulation about each element. The net result will be an altered pattern of turbulence. Each plane wave (Fourier component) Q_1, Q_2, $Q_3e^{i\underline{k}\cdot\underline{x}}$ will, in fact, be altered independently under the linearizing assumption to be made; the over-all effect on q_1, q_2, q_3 can be obtained by the summation expressed by equation (1a). It thus suffices to consider the effect of the contraction on a single representative plane wave.

EFFECT OF CONTRACTION ON REPRESENTATIVE PLANE WAVE

Velocity and vorticity at upstream station.—Designate by A a reference station upstream of the contraction and by B a reference station downstream of the contraction. (See fig. 1.) Let a typical Fourier component (plane wave) of the turbulent field $q_\alpha(\alpha=1, 2, 3)$ at station A be represented at time $t=0$ by

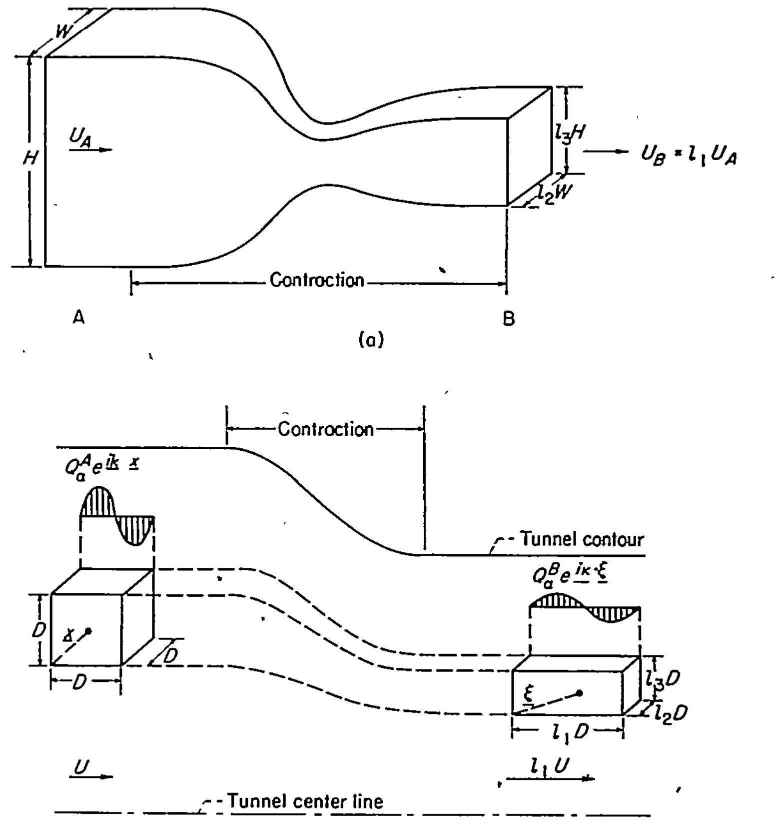
$$\tilde{q}_\alpha{}^A=\tilde{Q}_\alpha{}^Ae^{i(\underline{k}\cdot\underline{x})} \tag{8}$$

This wave, equation (8), is supposed to be carried along by the main stream with velocity U.

The vorticity $\tilde{\omega}_\alpha$ is obtained from the curl of equation (8) as

$$\tilde{\omega}_\alpha{}^A=i\sum_{\beta,\gamma}\epsilon_{\alpha\beta\gamma}k_\beta\tilde{Q}_\gamma{}^Ae^{i(\underline{k}\cdot\underline{x})} \tag{9}$$

where $\epsilon_{\alpha\beta\gamma}=\begin{cases}0, \text{ if any pair of subscripts are equal}\\ 1, \text{ if } \alpha\beta\gamma \text{ are in cyclic order}\\ -1, \text{ if } \alpha\beta\gamma \text{ are in anticyclic order}\end{cases}$



(a)

(b)

(a) Tunnel geometry.
(b) Stream tube geometry.
FIGURE 1.—Schematic representation of flow contraction parameters.

Distortion of fluid element in passage through contraction.—Suppose the contraction is such that the stream velocity U is increased by a factor l_1 between stations A and B while the breadth and the height of the tunnel are reduced by factors l_2 and l_3, respectively. (See fig. 1 (a).) In traveling from A to B, an initially cubical element of fluid of edge D will be distorted into a parallelepiped of edges l_1D, l_2D, l_3D (see fig. 1 (b)); a particle in the element originally ($t=0$) a vector distance $\underline{x}$ from a corner particle will finally ($t=t$) be found a distance $\underline{\xi}$ from the corner particle, where $\underline{\xi}$ is related to $\underline{x}$ by

$$\left.\begin{aligned}\xi_1&=l_1x_1\\ \xi_2&=l_2x_2\\ \xi_3&=l_3x_3\end{aligned}\right\} \tag{10}$$

In this argument the modification of the streamlines due to the turbulent velocity fluctuations has been neglected. This implies that the relative displacement of two adjacent particles due to the superposed turbulent motion is small compared with the displacement due to the tunnel contraction. This key assumption, due to Taylor (ref. 2), linearizes and vastly simplifies the problem. The limitations imposed by the assumption are discussed later under DECAY CONSIDERATIONS.

The velocity ratio l_1 and the lateral and vertical contraction ratios l_2 and l_3 are related by the continuity condition

$$\sigma l_1l_2l_3=1$$

where σ is the ratio of stream densities at stations B and A; the density is considered uniform at each station in accordance with the initial assumption of negligible turbulent density fluctuations.

Vorticity at downstream station.—The vorticity is carried along by the flow, the fluid elements undergoing the distortion pictured in figure 1 (b), to the approximation used. During the motion the strength changes in such a way as to maintain the constancy of circulation of the fluid elements.[5] The changes are expressed by the equations for the transport of vorticity in the Lagrangian form, due to Cauchy (ref. 8, p. 205).

$$\tilde{\omega}_\alpha{}^B=\sigma\sum_\beta\tilde{\omega}_\beta{}^A\frac{\partial\xi_\alpha}{\partial x_\beta}$$

where σ is the density ratio between stations B and A, and the derivatives $\partial\xi_\alpha/\partial x_\beta$ express the effect of the fluid distortion.[6] Evaluation by means of the distortion equations (10) yields simply

$$\tilde{\omega}_\alpha{}^B=\sigma l_\alpha\tilde{\omega}_\alpha{}^A$$

or, in expanded form

$$\left.\begin{aligned}\tilde{\omega}_1{}^B&=\sigma l_1\tilde{\omega}_1{}^A\\ \tilde{\omega}_2{}^B&=\sigma l_2\tilde{\omega}_2{}^A\\ \tilde{\omega}_3{}^B&=\sigma l_3\tilde{\omega}_3{}^A\end{aligned}\right\}\tag{11a}$$

These equations relating downstream and upstream vorticity embody the entire dynamics of the contraction effect. The equations are not limited to the plane sinusoidal waves discussed earlier, but apply to any (weak) vorticity distribution whatsoever.

The above derivation of the vorticity changes is substantially in the form given originally by G. I. Taylor (ref. 2) for the case $\sigma=1$ (incompressible flow). In order to assess the influence of the simplifying assumptions a more general derivation based on the Navier-Stokes equations is given in the section entitled DECAY CONSIDERATIONS.

By virtue of equation (9) as applied to equation (11a), the vorticity at station B is obtained explicitly as

$$\tilde{\omega}_\alpha{}^B=i\sigma l_\alpha\sum_{\beta,\gamma}\epsilon_{\alpha\beta\gamma}k_\beta\tilde{Q}_\gamma{}^A e^{i\underline{k}\cdot\underline{x}}\tag{11b}$$

where, it will be remembered, $\underline{x}$ is the radius vector to a fluid particle at time $t=0$ when the fluid element is at station A in the moving coordinate system of figure 1 (b). The corresponding vector to the particle at time $t=t$, when the fluid element is at station B, is $\underline{\xi}$ in that figure. When equations (10) are used to express $\underline{x}$ in terms of $\underline{\xi}$, the exponential $\underline{k}\cdot\underline{x}$ becomes, in expanded form,

$$\underline{k}\cdot\underline{x}=\frac{k_1\xi_1}{l_1}+\frac{k_2\xi_2}{l_2}+\frac{k_3\xi_3}{l_3}$$

The right-hand side may be expressed as $\underline{\kappa}\cdot\underline{\xi}$, where

$$\underline{\kappa}=\frac{k_1}{l_1},\frac{k_2}{l_2},\frac{k_3}{l_3}$$

defines a new wave-number vector.

Velocity at downstream station, general case.—The velocity distribution whose curl in the ξ_1, ξ_2, ξ_3 system is given by equation (11b) and which satisfies continuity is found to be expressible in the form

$$\tilde{q}_\alpha{}^B=\tilde{Q}_\alpha{}^B e^{i\underline{\kappa}\cdot\underline{\xi}}\tag{12}$$

with

$$\tilde{Q}_\alpha{}^B=\frac{1}{l_\alpha}\left(\tilde{Q}_\alpha{}^A-\sum_\beta\frac{\tilde{Q}_\beta{}^A k_\beta k_\alpha}{l_\beta{}^2\kappa^2}\right)\tag{13}$$

where κ is the magnitude of the wave-number vector $\underline{\kappa}$. This result is the general solution for the contraction effect on a single plane wave.

Equations (12) and (13) admit of a simple (but not obvious) geometrical interpretation: *traversal of the stream contraction alters the initial plane wave, equation (8), so that its wave-number vector $\underline{k}=(k_1, k_2, k_3)$ is transformed into $\underline{\kappa}=\frac{k_1}{l_1}, \frac{k_2}{l_2}, \frac{k_3}{l_3}$ and its amplitude vector $(\tilde{Q}_1{}^A, \tilde{Q}_2{}^A, \tilde{Q}_3{}^A)$ is transformed into the projection of $(\tilde{Q}_1{}^A/l_1, \tilde{Q}_2{}^A/l_2, \tilde{Q}_3{}^A/l_3)$ on a plane normal to the new wave-number vector $\underline{\kappa}$.*

Velocity at downstream station, axisymmetric contraction.—In case the stream contraction is axisymmetric,[7] a considerable simplification results. The condition for axisymmetry $l_2=l_3$, with use of the continuity equation (2), reduces equations (13) to

$$\left.\begin{aligned}\tilde{Q}_1{}^B&=\frac{\tilde{Q}_1{}^A}{l_1}\,\frac{k_1{}^2+k_2{}^2+k_3{}^2}{\epsilon k_1{}^2+k_2{}^2+k_3{}^2}\\ \tilde{Q}_2{}^B&=\frac{1}{l_2}\left[\tilde{Q}_2{}^A+\frac{\tilde{Q}_1{}^A k_1k_2(1-\epsilon)}{\epsilon k_1{}^2+k_2{}^2+k_3{}^2}\right]\\ \tilde{Q}_3{}^B&=\frac{1}{l_2}\left[\tilde{Q}_3{}^A+\frac{\tilde{Q}_1{}^A k_1k_3(1-\epsilon)}{\epsilon k_1{}^2+k_2{}^2+k_3{}^2}\right]\end{aligned}\right\}\tag{14}$$

where $\epsilon\equiv l_2{}^2/l_1{}^2$. The considerably greater complexity of equation (13) is perhaps obscured by the purposely expanded form of equations (14).

If the initial wave normal $\underline{k}$ is perpendicular to the (longitudinal) x_1-axis, the component k_1 vanishes and equations (14) reduce further to

$$\left.\begin{aligned}\tilde{Q}_1{}^B&=\tilde{Q}_1{}^A/l_1\\ \tilde{Q}_2{}^B&=\tilde{Q}_2{}^A/l_2\\ \tilde{Q}_3{}^B&=\tilde{Q}_3{}^A/l_2\end{aligned}\right\}\tag{15}$$

The same equations result when $\tilde{Q}_1{}^A$ may be neglected in comparison with $\tilde{Q}_2{}^A$ and $\tilde{Q}_3{}^A$, that is, when the amplitude vector is substantially normal to the x_1-axis. *Equations (15) state that an axisymmetric contraction defined by l_1, l_2 alters*

[5] This statement is exact for the postulated inviscid fluid. The modification produced by the diffusive effect of viscosity, in the case of a gas, becomes appreciable for the smaller eddies or higher wave numbers; for this analysis a criterion for neglect of viscous effects is $\nu k^2<<|dU/dx|$. (See DECAY CONSIDERATIONS, equation (43).)

[6] These equations refer to axes moving with some fluid particle rather than axes fixed as in reference 8; the form of the equations is unaffected.

[7] A contraction such that all cross sections of the tunnel are similar, whence $l_2=l_3$, is termed *axisymmetric; the sections need not be circular.*

these waves by a factor of $1/l_1$ *in the longitudinal velocity component and a factor of* $1/l_2$ *in the lateral velocity components. These equations apply only to particular types of wave; yet when the contraction effect is later integrated over the random aggregation of waves representing isotropic turbulence, the over-all results are found not to differ greatly from the simple factors* $1/l_1$ *and* $1/l_2$*, respectively.*

The same factors were obtained by Prandtl (ref. 1) for other special disturbances: the factor $1/l_1$ from energy considerations for purely longitudinal disturbance velocities, and the factor $1/l_2$ from momentum considerations for a rotating cylindrical element of fluid with axis normal to the stream.

EFFECT OF CONTRACTION ON SPECTRUM AND CORRELATION TENSORS

Effect of contraction on correlation tensor.—The analysis herein leads first to the changes in the spectrum tensor $\Gamma_{\alpha\beta}{}^A(\underline{k})\rightarrow\Gamma_{\alpha\beta}{}^B(\underline{\kappa})$. Then the corresponding changes in the correlation tensor may be obtained from the Fourier transform relation, equation (6b):

$$R_{\alpha\beta}{}^A(\underline{r})=\int\int_{-\infty}^{\infty}\int\Gamma_{\alpha\beta}{}^A(k)e^{-i\underline{k}\cdot\underline{r}}d\tau(\underline{k}) \tag{16a}$$

$$R_{\alpha\beta}{}^B(\underline{r})=\int\int_{-\infty}^{\infty}\int\Gamma_{\alpha\beta}{}^B(\underline{\kappa})e^{i\underline{\kappa}\cdot\underline{r}}d\tau(\underline{\kappa}) \tag{16b}$$

In succeeding paragraphs $\Gamma_{\alpha\beta}{}^B(\underline{\kappa})$ will be determined in terms of the initial spectrum tensor $\Gamma_{\alpha\beta}{}^A(\underline{k})$ for various cases.

Spectrum tensors at upstream and downstream stations in terms of Q_α.—In an earlier discussion, the Fourier coefficients Q_α were chosen so as to define a field of turbulence confined to a large parallelepiped of volume τ, and vanishing everywhere outside; for this case $\lim\limits_{\tau\to\infty}\frac{8\pi^3}{\tau}Q_\alpha Q_\beta{}^*$ was to be identified with the correlation tensor $\Gamma_{\alpha\beta}$. For station A upstream of the contraction it will be convenient to specialize this parallelepiped to a cube of edge D. Such a cube will, however, be distorted into a parallelepiped of edges l_1D, l_2D, l_3D by the stream contraction by the time it reaches station B downstream. (See fig. 1(b).) The spectrum tensors for stations A and B, respectively, are therefore

$$\left.\begin{aligned}\Gamma_{\alpha\beta}{}^A(\underline{k})&=\lim_{D\to\infty}\frac{8\pi^3}{D^3}Q_\alpha{}^A(\underline{k})Q_\beta{}^{A*}(\underline{k})\\ \Gamma_{\alpha\beta}{}^B(\underline{\kappa})&=\lim_{D\to\infty}\frac{8\pi^3}{l_1l_2l_3D^3}Q_\alpha{}^B(\underline{\kappa})Q_\beta{}^{B*}(\underline{\kappa})\end{aligned}\right\} \tag{17}$$

Evaluation of spectrum tensor at downstream station, general case.—The identifications made in the last paragraph allow the spectrum tensor to be evaluated at station B in terms of the spectrum tensor at station A and the parameters l_1, l_2, l_3 defining the stream contraction between stations A and B. For a single plane wave $\tilde{q}_\alpha{}^A=\tilde{Q}_\alpha{}^A e^{i\underline{k}\cdot\underline{x}}$, which is transformed by the contraction into $\tilde{q}_\alpha{}^B=\tilde{Q}_\alpha{}^B e^{i\underline{\kappa}\cdot\underline{\xi}}$, equations (13) give

$$\tilde{Q}_\alpha{}^B=\frac{1}{l_\alpha}\left(\tilde{Q}_\alpha{}^A-\sum_\beta\frac{\tilde{Q}_\beta{}^Ak_\beta k_\alpha}{l_\beta{}^2\kappa^2}\right)$$

In the Fourier integral representation $\tilde{q}_\alpha{}^A$ is to be interpreted as $dq_\alpha{}^A$, $\tilde{q}_\alpha{}^B$ as $dq_\alpha{}^B$, $\tilde{Q}_\alpha{}^A$ as $Q_\alpha{}^Ad\tau(\underline{k})$, and $\tilde{Q}_\alpha{}^B$ as $Q_\alpha{}^Bd\tau(\underline{\kappa})$. Accordingly,

$$Q_\alpha{}^B=\frac{l_1l_2l_3}{l_\alpha}\left(Q_\alpha{}^A-\sum_\beta\frac{Q_\beta{}^Ak_\beta k_\alpha}{l_\beta{}^2\kappa^2}\right) \tag{18}$$

since $l_1l_2l_3=d\tau(\underline{k})/d\tau(\underline{\kappa})$. Thus

$$Q_\alpha{}^BQ_\beta{}^{B*}=\frac{l_1{}^2l_2{}^2l_3{}^2}{l_\alpha l_\beta}\left[Q_\alpha{}^AQ_\beta{}^{A*}+\sum_{\gamma,\delta}\left(-\frac{Q_\alpha{}^AQ_\gamma{}^{A*}k_\gamma k_\beta}{l_\gamma{}^2\kappa^2}-\frac{Q_\beta{}^{A*}Q_\gamma{}^Ak_\gamma k_\alpha}{l_\gamma{}^2\kappa^2}+\frac{Q_\gamma{}^AQ_\delta{}^{A*}k_\gamma k_\delta k_\alpha k_\beta}{l_\gamma{}^2l_\delta{}^2\kappa^4}\right)\right]$$

The corresponding relation between the postcontraction and precontraction spectrum tensors is, by virtue of equations (17),

$$\Gamma_{\alpha\beta}{}^B(\underline{\kappa})=\frac{l_1l_2l_3}{l_\alpha l_\beta}\left[\Gamma_{\alpha\beta}{}^A(\underline{k})+\sum_{\gamma,\delta}\left(-\frac{\Gamma_{\alpha\gamma}{}^A(\underline{k})k_\gamma k_\beta}{l_\gamma{}^2\kappa^2}-\frac{\Gamma_{\gamma\beta}{}^A(\underline{k})k_\gamma k_\alpha}{l_\gamma{}^2\kappa^2}+\frac{\Gamma_{\gamma\delta}{}^A(\underline{k})k_\gamma k_\delta k_\alpha k_\beta}{l_\gamma{}^2l_\delta{}^2\kappa^4}\right)\right] \tag{19}$$

where $\underline{k}$ is related to $\underline{\kappa}$ by

$$k_1,\ k_2,\ k_3=l_1\kappa_1,\ l_2\kappa_2,\ l_3\kappa_3 \tag{20}$$

Special case: axisymmetric contraction but arbitrary initial spectrum.—When the contraction is axisymmetric ($l_2=l_3$), the equation of continuity in the form

$$\sum_\gamma k_\gamma\Gamma_{\gamma\delta}{}^A=0 \tag{21}$$

may be used to simplify equation (19). The result may be written

$$\Gamma_{\alpha\beta}{}^B(\underline{\kappa})=\frac{l_1l_2{}^2}{l_\alpha l_\beta}\left[\Gamma_{\alpha\beta}{}^A(\underline{k})+\frac{(\Gamma_{\alpha1}{}^Ak_\beta+\Gamma_{1\beta}{}^Ak_\alpha)k_1(1-\epsilon)}{\epsilon k_1{}^2+k_2{}^2+k_3{}^2}+\frac{\Gamma_{11}{}^Ak_1{}^2k_\alpha k_\beta(1-\epsilon)^2}{(\epsilon k_1{}^2+k_2{}^2+k_3{}^2)^2}\right] \tag{22}$$

where the ratio $\epsilon\equiv l_2{}^2/l_1{}^2$; for a large speed gain in the contraction $\epsilon\ll1$.

Special case: axisymmetric contraction, isotropic initial spectrum.—A further simplification occurs when the turbulence at station A is isotropic. In that case, the spectrum tensor $\Gamma_{\alpha\beta}{}^B(\underline{\kappa})$ downstream of the contraction can be expressed explicitly to within an unknown multiplicative factor $G(k)$. This results from the fact (ref. 3) that $\Gamma_{\alpha\beta}{}^A(\underline{k})$ must then be an isotropic second-order tensor; the isotropic property together with the continuity condition, equation (21), requires $\Gamma_{\alpha\beta}{}^A(\underline{k})$ to be of the form

$$\Gamma_{\alpha\beta}{}^A(\underline{k})=G(k)\ (k^2\delta_{\alpha\beta}-k_\alpha k_\beta) \tag{23}$$

where

$$\delta_{\alpha\beta}=\begin{cases}1\text{ for }\alpha=\beta\\0\text{ for }\alpha\neq\beta\end{cases}$$

The right-hand side of equation (22) may be evaluated by means of equation (23). The diagonal terms reduce to relatively simple forms:

$$\Gamma_{11}{}^B(\underline{\kappa})=\frac{l_2^2}{l_1}G(k)\frac{(k^2-k_1^2)k^4}{(\epsilon k_1^2+k_2^2+k_3^2)^2} \tag{24}$$

$$\Gamma_{22}{}^B(\underline{\kappa})=l_1\,G(k)\left[k^2-k_2^2-\frac{2k_1^2k_2^2(1-\epsilon)}{\epsilon k_1^2+k_2^2+k_3^2}+\frac{k_1^2k_2^2(k^2-k_1^2)(1-\epsilon)^2}{(\epsilon k_1^2+k_2^2+k_3^2)^2}\right] \tag{25}$$

and $\Gamma_{33}{}^B(\underline{\kappa})$ is obtained from $\Gamma_{22}{}^B(\underline{\kappa})$ by replacing k_2 by k_3 and vice versa. The relation, equation (20), between $\underline{k}$ and $\underline{\kappa}$ applies here.

One-dimensional longitudinal spectrums.—If the form of the intitial spectrum tensor $\Gamma_{\alpha\beta}{}^A(\underline{k})$ is known, the corresponding one-dimensional spectrums $F_\alpha{}^A(k_1)$ can be calculated, according to the defining equation (7) as applied at the upstream station A:

$$F_\alpha{}^A=2\int\int_{-\infty}^{\infty}\Gamma_{\alpha\alpha}{}^A(k_1,k_2,k_3)dk_2dk_3 \tag{26}$$

A particular case of isotropic turbulence is of special interest (ref. 4): In equation (23) for $\Gamma_{\alpha\beta}{}^A(\underline{k})$ the function $G(k)$ is taken to be $N(k^2+\gamma^2)^{-3}$, where N,γ are constants. Then

$$\Gamma_{11}{}^A=\frac{N(k_2^2+k_3^2)}{(k_1^2+k_2^2+k_3^2+\gamma^2)^3}$$

and after integration

$$F_1{}^A(k_1)=\frac{\pi N}{(k_1^2+\gamma^2)}$$

This one-dimensional longitudinal spectrum is of the same form as an empirical relation obtained in reference 5 for that of isotropic turbulence in the initial period; this agreement is the special virtue of the form assumed for $G(k)$.

The one-dimensional lateral spectrum functions corresponding to the same $G(k)$ are readily evaluated; they are

$$F_2{}^A(k_1)=F_3{}^A(k_1)=\frac{\pi N(3k_1^2+\gamma^2)}{2(k_1^2+\gamma^2)^2}$$

The equality of the F_2 and the F_3 functions results, of course, from the isotropy of the turbulence.

The effect of the stream contraction on these one-dimensional spectrums is found by employing the postcontraction value of $\Gamma_{\alpha\alpha}$, that is, the value $\Gamma_{\alpha\alpha}{}^B$ appropriate to the downstream station B. Since $\Gamma_{\alpha\alpha}{}^B$ is a function of the local wave-number vector $\underline{\kappa}$ at station B, the equation corresponding to (26) is

$$F_\alpha{}^B=2\int\int_{-\infty}^{\infty}\Gamma_{\alpha\alpha}{}^B(\kappa_1,\kappa_2,\kappa_3)d\kappa_2d\kappa_3$$

which is a function of κ_1.

For performing the integration and making later comparisons of spectrums, it is convenient to transform from κ_1, κ_2, κ_3 to k_1, k_2, k_3, where

$$\kappa_1=k_1/l_1$$
$$\kappa_2=k_2/l_2$$
$$\kappa_3=k_3/l_3$$

and to define $F_\alpha{}^B(k_1)=l_1^{-1}F_\alpha{}^B$, such that $\int_0^\infty F_\alpha{}^B(k_1)dk_1=\int_0^\infty F_\alpha{}^Bd\kappa_1$; thus

$$F_\alpha{}^B(k_1)=\frac{2}{l_1l_2l_3}\int\int_{-\infty}^{\infty}\Gamma_{\alpha\alpha}{}^B\left(\frac{k_1}{l_1},\frac{k_2}{l_2},\frac{k_3}{l_3}\right)dk_2dk_3 \tag{27}$$

The spectrum tensor elements $\Gamma_{\alpha\alpha}{}^B$ following an axisymmetric contraction have been evaluated in equations (24) and (25). With these values inserted and $G(k)$ specified as before, the integrations of equation (25) are best effected in polar coordinates. The results are expressed most simply in terms of a 'normalized' longitudinal wave number k_1/γ as incorporated in the two parameters

$$s\equiv\frac{k_1^2}{\gamma^2}+1$$

$$t\equiv\frac{\epsilon k_1^2-k_1^2}{\gamma^2}-1$$

The final result for the one-dimensional longitudinal spectrum following an axisymmetric contraction ($l_2=l_3$) is

$$F_1{}^B(k_1)=\frac{2\pi N}{l_1^2\gamma^2t^3}\left[3+4t+t^2+\frac{t}{2s}+\left(2+4s+2t+st+\frac{3s}{t}\right)\ln\left(\frac{s}{s+t}\right)\right] \tag{28}$$

The corresponding result for the one-dimensional lateral spectrums following an axisymmetric contraction is

$$F_2{}^B(k_1)=F_3{}^B(k_1)=\frac{\pi N}{2\gamma^2l_2^2t^2}\left(\frac{(3s-2)t^2}{s^2}-4(1-\epsilon)(s-1)\left\{\frac{2s+t}{2s}+\frac{s+t}{t}\ln\left(\frac{s}{s+t}\right)+\frac{1-\epsilon}{2t}\left[\frac{6s+5t}{2}+\frac{(s+t)(3s+t)}{t}\ln\left(\frac{s}{s+t}\right)\right]\right\}\right) \tag{29}$$

and for $\epsilon\ll1$ (large speed gain) a simple but very close approximation is

$$F_2{}^B(k_1)=\frac{\pi N}{2l_2^2\gamma^2}\frac{1+2k_1^2/\gamma^2}{(1+k_1^2/\gamma^2)^2}+0(\epsilon,\epsilon\ln\epsilon)$$

(The corresponding approximation for $F_1{}^B(k_1)$ is not simple enough to warrant noting.) The parameters l_1, $l_2=l_3$, and $\epsilon=l_2^2/l_1^2$ are related to the initial and final Mach numbers of the main stream, if the fluid is air, by the equations

$$\left.\begin{aligned}l_1^2&=\left(\frac{M_B}{M_A}\right)^2\left(\frac{5+M_A^2}{5+M_B^2}\right)\\ l_2^2&=\frac{M_A}{M_B}\left(\frac{5+M_B^2}{5+M_A^2}\right)^3\\ \epsilon&=\left(\frac{M_A}{M_B}\right)^3\left(\frac{5+M_B^2}{5+M_A^2}\right)^4\end{aligned}\right\} \tag{30a}$$

For incompressible flow (M_B, $M_A\rightarrow 0$) these reduce to

$$\left.\begin{aligned} l_1&=U_B/U_A\\ l_2&=l_1^{-1/2}\\ \epsilon&=l_1^{-3}\end{aligned}\right\} \tag{30b}$$

These postcontraction spectrums, equations (28) and (29), are compared with the initial spectrums in figures 2 and 3, respectively; the comparison is based on an assumed initial stream Mach number of 0.05 (station A) followed by an axisymmetric contraction such that the final Mach number is 2.0 (station B); the corresponding parameters are $l_1=29.8$, $l_2=0.382$, $\epsilon=0.00016$. Consider first the longitudinal spectrums, figure 2. Normalizing factors are incorporated such that the areas under the two curves, if replotted on a linear scale, would be the same; this normalization serves to differentiate changes in shape from changes in amplitude. The figure exhibits a rather striking distortion of the spectrum after traversing the stream contraction: the peak spectral density is shifted from zero wave number to $k_1/\gamma=1.4$ along with a general shift of density to the higher wave numbers. Associated with this change in shape is a reduction in amplitude by the factor $\overline{u_B^2}/\overline{u_A^2}$, $\overline{u_A^2}$ and $\overline{u_B^2}$ being the respective integrals of the spectral density curves. These integrals are evaluated in a later section.

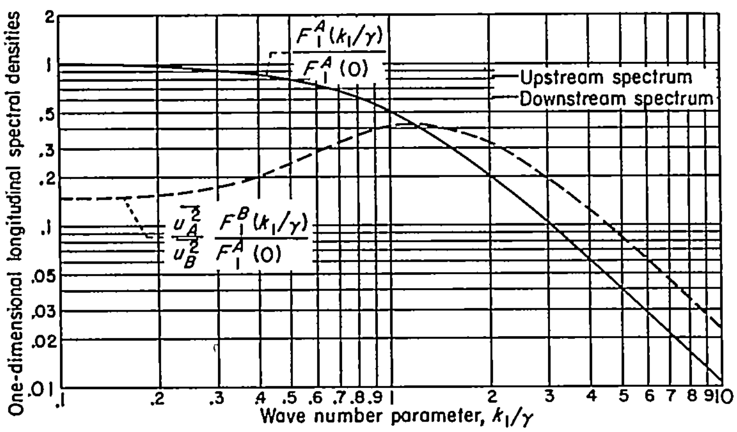


FIGURE 2.—Comparison of one-dimensional longitudinal spectrum upstream (F_1^A) and downstream (F_1^B) of axisymmetric contraction. Curves normalized to same area. Isotropic initial turbulence with $G(k)=N(k^2+\gamma^2)^{-3}$. $M_A=0.05$, $M_B=2.0$, corresponding to $l_1=29.821$, $\epsilon=0.0001637$.

The corresponding comparison for the lateral one-dimensional spectrums is made in figure 3. In this case the axisymmetric contraction has made very little distortion in the spectrum. There is again a change in magnitude (this time an increase) in the ratio $\overline{v_B^2}/\overline{v_A^2}$.

The changes in magnitude (that is, the changes in area under the spectral density curves) correspond to the changes $\overline{u_B^2}/\overline{u_A^2}$ and $\overline{v_B^2}/\overline{v_A^2}$ in the mean-square components of turbulence and are, at least qualitatively, well-known. The predicted changes in the shape of the spectrum curves are apparently new.

In the above comparisons both precontraction and postcontraction spectrums have been expressed in terms of the precontraction longitudinal wave number k_1, whereas the local postcontraction wave number is $\kappa_1=k_1/l_1$. Consider, however, a representative longitudinal wave which has the form $\cos k_1x$ at station A and $\cos \kappa_1\xi$ at station B. If x and ξ

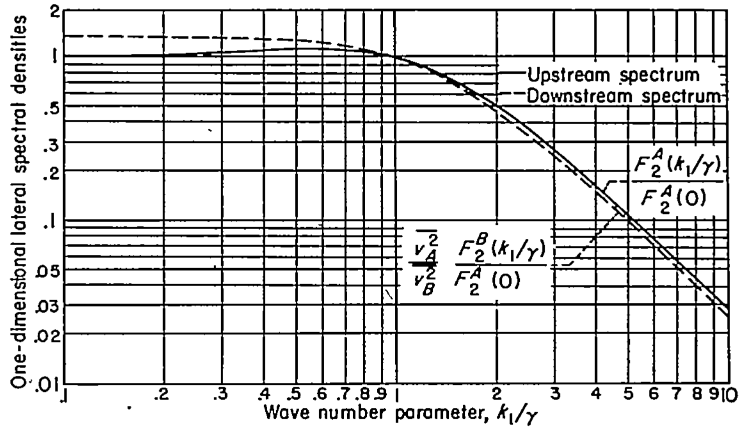


FIGURE 3.—Comparison of one-dimensional lateral spectrum upstream (F_2^A) and downstream (F_2^B) of axisymmetric contraction. Curves normalized to same area. Isotropic initial turbulence with $G(k)=N(k^2+\gamma^2)^{-3}$; $M_A=0.05$, $M_B=2.0$, corresponding to $l_1=29.821$ $\epsilon=0.0001637$.

are identified with the respective distances swept in time t by the moving waves over stationary hot-wire probes at stations A and B, respectively, then $k_1x=k_1U_At$ and $\kappa_1\xi=\kappa_1U_Bt=\frac{k_1}{l_1}l_1U_At$. Thus the (temporal) frequency seen by the hot wire in both cases is $k_1U_A/2\pi$. The comparison based on k_1 therefore constitutes, in effect, a comparison of the time spectrums that would be seen by stationary hot-wire probes, in contradistinction to the space spectrums discussed in the earlier parts of the paper.

EFFECT OF CONTRACTION ON MEAN-SQUARE VELOCITY COMPONENTS FOR ISOTROPIC TURBULENCE

The mean-square velocity components of the turbulent field may be identified as the diagonal terms of the correlation tensor $R_{\alpha\beta}(\underline{r})$ with $\underline{r}$ set equal to zero. Thus

$$\overline{u^2}=R_{11}(0)$$

$$\overline{v^2}=R_{22}(0)$$

$$\overline{w^2}=R_{33}(0)$$

where u, v, w have been written for q_1, q_2, q_3, respectively. The evaluation of these means is much less laborious than the evaluation of the general correlation tensor. In particular, the evaluation of the ratio of the means $\overline{u_B^2}/\overline{u_A^2}$, etc., may be made when the initial turbulence is specified to be isotropic but no further details of its spectrum are known. These ratios will be calculated in the following paragraphs.

Evaluations of $\overline{u^2}$ and $\overline{v^2}$ at upstream station.—According to equation (6)

$$R_{11}{}^A(0)=\overline{u_A^2}=\iiint_{-\infty}^{\infty}\Gamma_{11}{}^A(\underline{k})d\tau(\underline{k})$$

For isotropic turbulence $\Gamma_{\alpha\beta}$ has the form specified in equation (23), whence

$$\overline{u_A^2}=\iiint_{-\infty}^{\infty}G(k)(k^2-k_1^2)d\tau(\underline{k})$$

where $G(k)$ is an arbitrary function. It is convenient to transform to spherical polar coordinates:

$$\left.\begin{aligned} k_1&=k\cos\theta \\ k_2&=k\sin\theta\cos\varphi \\ k_3&=k\sin\theta\sin\varphi \\ d\tau(\underline{k})&=k^2\sin\theta\, d\theta\, d\varphi\, dk \end{aligned}\right\} \tag{31}$$

Then

$$\overline{u_A^2}=\int_0^\infty k^4G(k)dk\int_0^{2\pi}d\varphi\int_0^\pi \sin^3\theta\, d\theta \tag{32}$$

For the present purpose the function $G(k)$, which, together with the condition of isotropy, defines the turbulence, may be left unspecified; the integral involving $G(k)$ will cancel out in forming the ratio $\overline{u_B^2}/\overline{u_A^2}$. Let this integral have the value H; then

$$\overline{u_A^2}=\frac{8}{3}\pi H$$

By virtue of the assumed isotropy

$$\overline{v_A^2}=\overline{w_A^2}=\frac{8}{3}\pi H$$

Evaluation of ratio of $\overline{u^2}$ at downstream station to $\overline{u^2}$ at upstream station.—The mean value $\overline{u_B^2}$ is obtained from an integration involving the spectrum tensor after the latter has been transformed by passage of the flow through the tunnel contraction; according to equation (16b)

$$R_{11}^B(0)=\overline{u_B^2}=\iiint_{-\infty}^{\infty}\Gamma_{11}^B(\underline{\kappa})d\tau(\underline{\kappa})$$

For the present case, where the spectrum tensor at station A is assumed isotropic and the contraction is axisymmetric, the transformed tensor $\Gamma_{11}^B(\kappa)$ has been determined in equation (24). Thus

$$\overline{u_B^2}=\frac{l_2^2}{l_1}\iiint_{-\infty}^{\infty}\frac{G(k)k^4(k^2-k_1^2)d\tau(\underline{\kappa})}{(\epsilon k_1^2+k_2^2+k_3^2)^2}$$

Because of the unspecified function $G(k)$, it is convenient to change the variables of integration from the components of $\underline{\kappa}$ to the components of k. In other words, a transformation is made from the "wave-number space" of station B to the "wave-number space" of station A. The transformation follows from the Cartesian relations

$$\left.\begin{aligned} d\tau(\underline{k})&=dk_1dk_2dk_3 \\ d\tau(\underline{\kappa})&=d\kappa_1d\kappa_2d\kappa_3 \\ &=\frac{dk_1}{l_1}\frac{dk_2}{l_2}\frac{dk_3}{l_3} \end{aligned}\right\} \tag{33}$$

together with $l_2=l_3$ for an axisymmetric contraction, whence

$$\overline{u_B^2}=\frac{1}{l_1^2}\iiint_{-\infty}^{\infty}\frac{k^4G(k)(k^2-k_1^2)d\tau(\underline{k})}{(\epsilon k_1^2+k_2^2+k_3^2)^2}$$

Again the polar-coordinate transformation (equation (31)) is made, with the result

$$\overline{u_B^2}=\frac{1}{l_1^2}\int_0^\infty k^4G(k)dk\int_0^{2\pi}d\varphi\int_0^\pi\frac{\sin^3\theta\, d\theta}{(\epsilon\cos^2\theta+\sin^2\theta)^2}$$

The first two integrals occur also in $\overline{u_A^2}$ (equation (32)), and they cancel in obtaining the ratio $\overline{u_B^2}/\overline{u_A^2}$; thus

$$\frac{\overline{u_B^2}}{\overline{u_A^2}}=\frac{3}{4l_1^2}\int_0^\pi\frac{\sin^3\theta\, d\theta}{(\epsilon\cos^2\theta+\sin^2\theta)^2}$$

The final result may be written

$$\frac{\overline{u_B^2}}{\overline{u_A^2}}=\frac{3}{4l_1^2}\left[\frac{-1}{1-\epsilon}+\frac{2-\epsilon}{(1-\epsilon)^{3/2}}\tanh^{-1}\sqrt{1-\epsilon}\right] \tag{34}$$

and an asymptotic expansion for small ϵ is

$$\frac{\overline{u_B^2}}{\overline{u_A^2}}=-\frac{3}{4l_1^2}\left[1+\frac{3}{2}\epsilon+(1+\epsilon)\ln\frac{\epsilon}{4}+0(\epsilon^2\ln\epsilon)\right]$$

Equation (34) gives the ratio of the mean-square longitudinal velocity fluctuation downstream of an axisymmetric tunnel contraction to the corresponding mean square upstream of the contraction, when the initial turbulence is isotropic. The contraction is characterized by an increase in the stream speed in the specified ratio l_1 and a decrease in the lateral dimensions in the specified ratio l_2; the parameters l_1, l_2, and $\epsilon\equiv l_2^2/l_1^2$ are completely defined by the initial and final Mach numbers of the stream according to equations (30a) and (30b).

The variation of $\sqrt{\overline{u_B^2}/\overline{u_A^2}}$ with the speed ratio l_1 is plotted in figure 4 for two examples: in the first the flow is assumed compressible with a Mach number 0.05 at the start of the contraction; in the second the flow is assumed incompressible (M_A, $M_B\to 0$). The Mach number scale at the bottom applies only to the compressible case, the l_1 scale to both cases. The salient characteristic of the curves is the marked reduction in the longitudinal component of turbulence with increasing speed ratio l_1.

Compressibility is seen to have but a secondary effect, which is appreciable only at supersonic speeds. Note (equations (30) and (34)) that with l_1 as the independent variable, the effect of compressibility appears only in the parameter ϵ. The physical significance of ϵ follows from the definition of l_1 as the speed ratio provided by the contraction and l_2^2 as the area ratio of the contraction (in the axisymmetric case considered), with $\epsilon\equiv l_2^2/l_1^2$. For supersonic final speeds it is more proper to speak of a converging-diverging nozzle than a contraction, the term 'contraction' having been retained herein primarily for reasons of past usage.

The basis of the compressibility effect may be summed up in the following way. The influence of an axisymmetric stream contraction arises from distortion of the fluid elements, as described by the parameters l_1 and l_2. (See fig. 1 (b).)

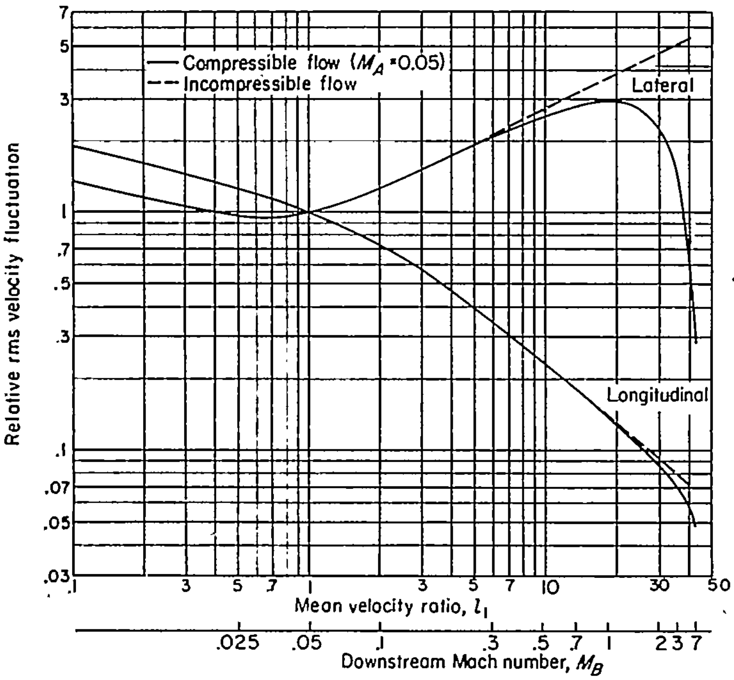


FIGURE 4.—Typical examples of selective effect of axisymmetric stream contraction without decay on components of turbulent intensity, showing influence of compressibility. Isotropic turbulence at $l_1=1$.

These parameters are related by the continuity condition $\sigma l_1 l_2^2=1$, where σ is the density ratio. Thus compressibility, in allowing σ to deviate from unity, changes the relation between l_1 and l_2 somewhat, and consequently modifies the contraction effect.

The graph of equation (34) in figure 4 is primarily for illustrative purposes; a form more useful for engineering applications is given in figure 5. The single curve provides the variation of $\sqrt{\overline{u_B^2}/\overline{u_A^2}}$ with both l_1 and ϵ; l_1 and ϵ may be determined from the initial and final Mach numbers by means of the simple relations (30a).

Evaluation of ratio of $\overline{v^2}$ at downstream station to $\overline{v^2}$ at upstream station.—The value of $\overline{v_B^2}$ results from an integration involving the transformed spectrum tensor, according to equation (16b)

$$R_{22}^B(0)=\overline{v_B^2}=\iiint_{-\infty}^{\infty}\Gamma_{22}^B(\underline{\kappa})d\tau(\underline{\kappa})$$

For isotropic initial turbulence and an axisymmetric contraction, the transformed spectrum tensor $\Gamma_{22}^B(\underline{\kappa})$ has been evaluated in equation (25). Thus

$$\overline{v_B^2}=l_1\iiint_{-\infty}^{\infty}G(k)\left[k^2-k_2^2-\frac{2k_1^2k_2^2(1-\epsilon)}{\epsilon k_1^2+k_2^2+k_3^2}+\frac{k_1^2k_2^2(k^2-k_1^2)(1-\epsilon)^2}{(\epsilon k_1^2+k_2^2+k_3^2)^2}\right]d\tau(\underline{k})$$

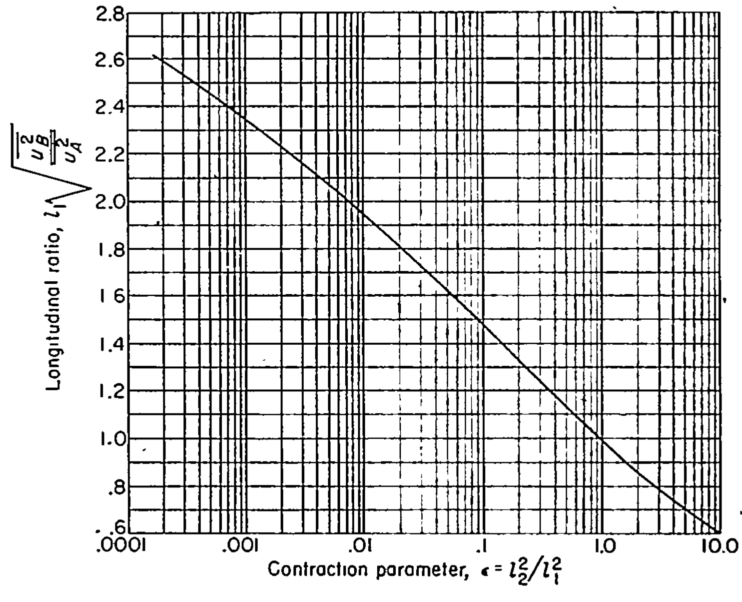


FIGURE 5.—Variation of relative root-mean-square longitudinal velocity fluctuation with both speed ratio l_1 and area ratio l_2^2 for axisymmetric contraction and isotropic turbulence at $l_1=1$.

Again it is convenient to transform from $\underline{\kappa}$-space to $\underline{k}$-space (equations (33)) and to introduce polar coördinates k, φ, and θ (equations (31)). The integrations with respect to k and φ are readily disposed of, with the result

$$\overline{v_B^2}=\frac{H}{l_2^2}\left[2\pi\int_0^\pi \sin^3\theta\,d\theta-2\pi(1-\epsilon)\int_0^\pi\frac{\sin^3\theta\cos^2\theta\,d\theta}{\sin^2\theta+\epsilon\cos^2\theta}+\pi(1-\epsilon)^2\int_0^\pi\frac{\sin^5\theta\cos^2\theta\,d\theta}{(\sin^2\theta+\epsilon\cos^2\theta)^2}\right]$$

where $H\equiv\int_0^\infty k^4G(k)dk$, as before. Upon carrying out the integration and dividing by $\overline{v_A^2}=\frac{8}{3}\pi H$, there is obtained finally

$$\frac{\overline{v_B^2}}{\overline{v_A^2}}=\frac{3}{8l_2^2}\left[\frac{2-\epsilon}{1-\epsilon}-\frac{\epsilon^2}{(1-\epsilon)^{3/2}}\tanh^{-1}\sqrt{1-\epsilon}\right]\tag{35}$$

For small ϵ this has the asymptotic expansion

$$\frac{\overline{v_B^2}}{\overline{v_A^2}}=\frac{3}{8l_2^2}\left\{\left[2+\epsilon+\frac{1}{2}\epsilon^2\ln\epsilon\right]+0(\epsilon^2)\right\}$$

Equation (35) gives the ratio of the mean-square lateral velocity fluctuation downstream of an axisymmetric tunnel contraction to the corresponding mean square upstream of the contraction, where the turbulence has been assumed to be isotropic. The variation of $\sqrt{\overline{v_B^2}/\overline{v_A^2}}$ with the speed ratio l_1 is plotted in figure 4, which already contains the graph of $\sqrt{\overline{u_B^2}/\overline{u_A^2}}$ discussed earlier; again the two cases are incompressible flow and compressible flow with an initial Mach number of 0.05. For $l_1\geq1$ and incompressible flow, the lateral component of turbulence is seen to increase steadily with l_1, in marked contrast to the decrease exhibited by the

longitudinal component. The curve (of the lateral component) for compressible flow begins to differ sensibly from the curve for incompressible flow for downstream Mach numbers above 0.3; above sonic speed compressibility is seen to effect a complete reversal of the curve. The over-all effect of compressibility on the contraction effect is thus much greater for the lateral than for the longitudinal component of the turbulence.

The graph of equation (35) in figure 4 is primarily illustrative; a form more useful for engineering applications is given in figure 6. The single curve provides the variation of $\sqrt{v_B^2/v_A^2}$ with both l_1 and ϵ; l_1 and ϵ may be determined from the initial and final Mach numbers by means of equations (30a).

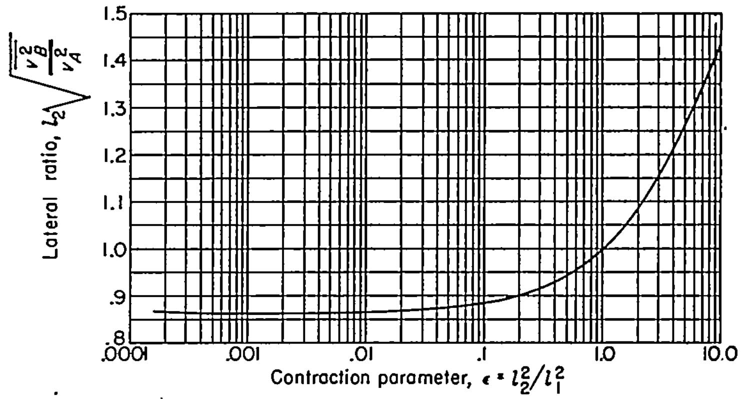


FIGURE 6.—Variation of relative root-mean-square lateral velocity fluctuation with both speed ratio l_1 and area ratio l_2^2 for axisymmetric contraction and isotropic turbulence at $l_1=1$.

DECAY CONSIDERATIONS

CRITERIA FOR NEGLIGIBLY SMALL DECAY

The basis of the present analysis of the contraction effect is embodied in equations (11a) relating the precontraction and postcontraction vorticity distributions. The simplicity of this result and its derivation arises from the neglect of the turbulent decay; by decay is meant the viscous dissipation and all the (nonlinear) intermixing processes of the eddies which together cause the mean turbulent intensity to diminish with time. The postulation of an inviscid fluid eliminated the viscous dissipation, and the limitation to very weak turbulence eliminated the intermixing processes. (While there can be no dissipation in an inviscid fluid, the intermixing processes ordinarily associated with decay will occur.) In order to assess the influence of these assumptions, equations (11a) will now be derived in a more general fashion with the Navier-Stokes equations as the starting point. For simplicity the fluid is taken to be incompressible, since the major conclusions are unaffected thereby.

General formulation of changes in vorticity.—By rearrangement and cross differentiation to eliminate the pressure term (ref. 8, p. 578), the Navier-Stokes equations can be transformed into

$$\frac{D\omega_1}{Dt}=\omega_1\frac{\partial q_1'}{\partial x_1}+\omega_2\frac{\partial q_1'}{\partial x_2}+\omega_3\frac{\partial q_1'}{\partial x_3}+\nu\nabla^2\omega_1 \tag{36}$$

and two similar equations, where $\underline{\omega}=\omega_1,\ \omega_2,\ \omega_3$ is the vorticity and $\underline{q}'=q_1',\ q_2',\ q_3'$ is the resultant velocity. Now let $\underline{q}'$ be the sum of a stream velocity $U,\ V,\ W$ and a turbulent velocity field $\underline{q}=q_1,\ q_2,\ q_3$; also, let curl $U,\ V,\ W=0$, so that $\underline{\omega}$ is just curl $\underline{q}$. Then equation (36) becomes (in tensor notation)

$$\frac{D\omega_1}{Dt}=\underbrace{\omega_\beta\frac{\partial U}{\partial x_\beta}}_{\text{Contraction}}+\underbrace{\omega_\beta\frac{\partial q_1}{\partial x_\beta}+\nu\nabla^2\omega_1}_{\text{Decay}} \tag{37}$$

and there are two similar equations. The first set of terms on the right-hand side is identified as the contraction effect, the second set as the decay effect. First the decay terms will be neglected in an attempt to recover equations (11a); then the neglected decay terms will be examined and criteria for their neglect arrived at.

Neglect of decay terms.—Equation (37) minus the viscous terms reads, in expanded form,

$$\frac{D\omega_1}{Dt}=\omega_1\frac{\partial(U+q_1)}{\partial x_1}+\omega_2\frac{\partial(U+q_1)}{\partial x_2}+\omega_3\frac{\partial(U+q_1)}{\partial x_3} \tag{38}$$

In this and the earlier equations $\frac{D}{Dt}$ is the 'Lagrangian' derivative following the fluid motion. Now consider a line segment $\delta x_1,\ \delta x_2,\ \delta x_3$ following the fluid motion: its Lagrangian derivative can be shown to be

$$\frac{D\delta x_1}{Dt}=\delta x_1\frac{\partial(U+q_1)}{\partial x_1}+\delta x_2\frac{\partial(U+q_1)}{\partial x_2}+\delta x_3\frac{\partial(U+q_1)}{\partial x_3} \tag{39}$$

and two similar equations. It can be seen that a solution of equations (38) and (39), together with their companion equations, is given by

$$\omega_1,\ \omega_2,\ \omega_3\sim\delta x_1,\ \delta x_2,\ \delta x_3$$

for all time t; this result is well known. Now complete the neglect of the decay terms by omitting the terms in q_1 in equation (38) and correspondingly in equation (39). By this neglect the turbulent perturbations of the flow streamlines have been suppressed: this can be inferred from the revised equation (39). If the particles are at station A at a time $t=0$ and reach station B at time $t=t$, there results

$$\frac{\omega_1^B}{\omega_1^A}=\frac{\delta x_1^B}{\delta x_1^A}$$

and two similar equations. But $\frac{\delta x_1^B}{\delta x_1^A}$ is just l_1, $\frac{\delta x_2^B}{\delta x_2^A}$ is l_2, and $\frac{\delta x_3^B}{\delta x_3^A}$ is l_3.

Therefore equations (11a) have been recovered for the incompressible case (density ratio $\sigma=1$).

Consideration of inertial decay terms.—In equations (37) to (39) the decay terms not involving ν are the inertial or

intermixing terms. These are seen to be nonlinear. The condition for their neglect is evidently

$$\left|\omega_\beta \frac{\partial q_1}{\partial x_\beta}\right| \ll \left|\omega_\beta \frac{\partial U}{\partial x_\beta}\right| \tag{40}$$

and two similar conditions between q_2 and V, q_3 and W, respectively. In a contraction like that of a wind tunnel, the dominant velocity gradients will be $\frac{\partial U}{\partial x_1}$, $\frac{\partial V}{\partial x_2}$, $\frac{\partial W}{\partial x_3}$, and these will be of the same order of absolute magnitude. A sufficient condition to replace expression (40) is therefore

$$\left|\frac{\partial q_\alpha}{\partial x_\beta}\right| \ll \left|\frac{\partial U}{\partial x_1}\right| \tag{41}$$

that is, all of the turbulent velocity gradients are very much less than the axial gradient of the stream velocity. This is essentially the assumption underlying the distortion equations (10), which led directly to the vorticity changes (11a) in Taylor's method.

In statistical terms an approximate inference from equation (41) is, for isotropic turbulence,

$$\sqrt{\overline{\left(\frac{\partial u}{\partial x_1}\right)^2}} \ll \left|\frac{\partial U}{\partial x_1}\right|$$

But by definition of λ this may be written

$$\frac{\sqrt{\overline{u^2}}}{\lambda} \ll \left|\frac{\partial U}{\partial x_1}\right| \tag{42}$$

The "microscale" λ may be interpreted as a sort of average eddy diameter weighted in favor of the smaller eddies. Equation (42) may be accepted as a practical criterion for the neglect of the inertial decay terms, equivalent to one of the two assumptions underlying equation (10). The other assumption, neglect of viscosity, is considered next.

Consideration of viscous decay term.—The viscous decay term in equation (37) is the term containing ν. This term is linear and so will affect individual plane waves separately without mutual interference. The magnitude of the term may be estimated to a sufficient approximation by considering a wave carried along by the contracting stream and neglecting (for this term only) the distortion of the wave imposed by the contraction. Thus a component of the wave may be written

$$\omega_1 = \Omega_1 e^{i(\underline{k}\cdot\underline{x} - k_1 U t)}$$

Then, if the inertial decay terms of equation (37) are negligible, the equation reads, with $\left|\frac{\partial U}{\partial x_2}\right| = \left|\frac{\partial U}{\partial x_3}\right| \ll \left|\frac{\partial U}{\partial x_1}\right|$

$$\begin{aligned}\frac{D\omega_1}{Dt} &= \omega_1 \frac{\partial U}{\partial x_1} + \nu\nabla^2\omega_1 \\ &= \omega_1 \frac{\partial U}{\partial x_1} + \nu(k_1^2 + k_2^2 + k_3^2)\omega_1 \\ &= \omega_1 \left(\frac{\partial U}{\partial x_1} + \nu k^2\right)\end{aligned}$$

Accordingly, viscosity may be neglected for that portion of the spectrum which satisfies the inequality

$$\nu k^2 \ll \left|\frac{\partial U}{\partial x_1}\right| \tag{43}$$

In the "initial period" of decay, if the inertial decay criterion (42) is satisfied, the major part of the spectrum will satisfy condition (43).

ROUGH ESTIMATION OF MUTUAL EFFECTS OF DECAY AND CONTRACTION

When the decay effects are not negligible compared with the contraction effects (see criterion developed in the last section) the theoretical basis of the present theory of the contraction effect is violated. Because negligible decay is more the exception than the rule, there is considerable incentive to attempt to apply the theory outside the valid range by means of assumptions concerning the simultaneous effects of decay and contraction.

Suppose, now, the decay and the contraction are considered to occur alternatively in small steps, starting from isotropic turbulence. Each stream tube is considered to contract stepwise: between steps there is decay without contraction; at each step there is a sudden contraction without decay. Let the change in speed ratio per step be dl_1, the reduction in $\overline{u^2}$ due to decay be $(\overline{du^2})_D$, and the reduction in $\overline{u^2}$ due to contraction be $(\overline{du^2})_C$. Express the effect of decay in the absence of contraction in the form

$$\left(\frac{\overline{u^2}}{\overline{u_A{}^2}}\right)_D = D(l_1) \tag{44}$$

where l_1 is a function of the time of travel (decay time) t, and the effect of contraction in the absence of decay in the form

$$\left(\frac{\overline{u^2}}{\overline{u_A{}^2}}\right)_C = C(l_1) \tag{45}$$

The corresponding differential forms are

$$\frac{(\overline{du^2})_D}{\overline{u^2}} = \frac{D'(l_1)}{D(l_1)}\, dl_1 \tag{46}$$

$$\frac{(\overline{du^2})_C}{\overline{u^2}} = \frac{C'(l_1)}{C(l_1)}\, dl_1 \tag{47}$$

The assumption is now made that equation (46) applies to the decay effect per step and equation (47) to the contraction effect per step, the only interaction being in the common $\overline{u^2}$.[8] This assumption neglects the tendency of the decay process to counteract the anisotropy introduced by the

[8] It is known that in the 'initial' period of decay $\frac{d\overline{u^2}}{\overline{u^2}} \sim -\overline{u^2}dt$. Equation (46) amounts to replacing the $-\overline{u^2}$ on the right-hand side by $-\overline{(u^2)}_{\text{decay only}}$; some defense may be made of this approximation, considering the progressive deviation from isotropy.

contraction effect. The total effect per step is then

$$\frac{d\overline{u^2}}{\overline{u^2}}=\left[\frac{C'(l_1)}{C(l_1)}+\frac{D'(l_1)}{D(l_1)}\right]dl_1$$

whence upon integration the over-all effect is

$$\frac{\overline{u_B^2}}{\overline{u_A^2}}=C(l_1)D(l_1) \tag{48}$$

That is, if the effect of contraction alone is expressed by $C(l_1)$ (equation (45)) and the effect of decay alone by $D(l_1)$ (equation (44)), then the joint effect under the assumption is expressed by the product $C(l_1)\,D(l_1)$.

Equation (48) is intended to provide a very rough adjustment of the theoretical contraction effect $C(l_1)$ to account for decay. This adjustment will be made in the attempt to compare the theory with experimental results in which the decay effects are of the same order as the contraction effects.

Equation (48) refers to the longitudinal velocity component u; an equation of the same form is obtained for the lateral component v. For both cases the function $D(l_1)$ is taken to be the right-hand side of the empirical decay law (ref. 9)

$$\left(\frac{\overline{u_B^2}}{\overline{u_A^2}}\right)_D=\frac{1}{1+0.58\dfrac{\sqrt{\overline{u_A^2}}}{L_A}\,t(l_1)}=D(l_1) \tag{49}$$

for isotropic turbulence in the initial period. The decay time $t(l_1)$ in the formula is the time required by a particle of the main stream to pass through the contraction, the initial velocity being U_A and the final velocity l_1U_A.

COMPARISON WITH EXPERIMENT

There appear to have been no experimental investigations with which to compare the predicted changes imposed by a stream contraction on the spectrum of the turbulence, or on the correlation tensor of the turbulence. The available experimental data seem to be limited to measurements bearing on the changes in the root-mean-square velocity components. These data apply, moreover, to conditions outside the proper scope of the present theory in that large decay effects are present. The experimental data are therefore compared with a crude extension of the theory in which the decay is allowed for in first approximation. (See preceding section.)

The most extensive data are those of MacPhail (ref. 10), which in effect cover a range of contraction ratios from $l_1=1$ to $l_1=9.65$ inasmuch as measurements were made at various stations along the contraction. Isolated points for particular contraction ratios were obtained from investigations made for other purposes by Dryden and Schubauer (ref. 11) and by Hall (ref. 12). Only those points were chosen for which the initial turbulence was indicated to be approximately isotropic.

In table I are listed, for the three experimental arrangements, the parameters used in the estimation of the decay factor (equations (48) and 49)). In reference 10 the initial stream velocity U_A and scale of turbulence L_A were given. In reference 11 the value of U_A was given, and the value of L_A was taken to be 0.05 foot, the only scale mentioned; it was not clear, however, whether this value of scale applied with or without screens. In reference 12 the value of U_A was inferred from collateral information and is somewhat uncertain; the scale L_A was estimated from the dimensions of the honeycomb. In all three experimental arrangements the initial relative levels of turbulence were specified. The decay time t of the turbulence was computed as the time for a particle to traverse the contraction; the value arrived at for Hall's data (ref. 12) reflects the uncertainty in the assumed U_A.

Root-mean-square longitudinal velocity components.—The comparison of the theory, including estimated decay, with experiment for the longitudinal component of turbulence is given in figure 7. The theoretical curve, in each instance, is the product of a value computed for contraction alone, neglecting decay, (obtainable from fig. 5) and a value estimated for decay alone neglecting contraction. (See equations (48) and (49).) The agreement with MacPhail's data and with Hall's single point can be considered good. The agreement with the Dryden-Schubauer point, on the other hand, is poor; a slight improvement would result on correction for the spurious contribution of the noise background.

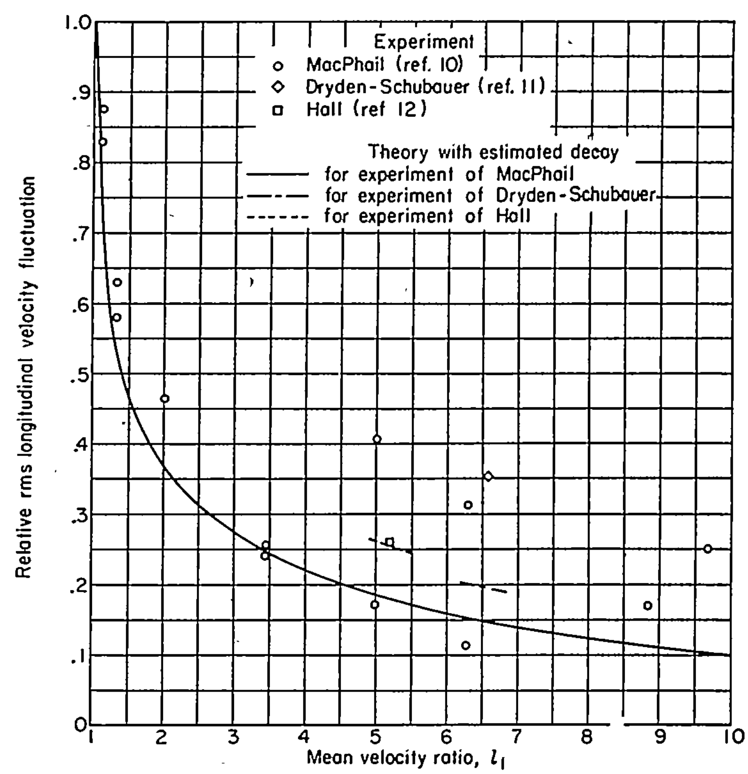


FIGURE 7.—Comparison of predicted axisymmetric contraction effect with experiment for longitudinal component of turbulence, with decay allowed for in first approximation. Initial isotropic turbulence assumed.

Root-mean-square lateral velocity components.—Comparison of the theory, again including estimated decay, with experiment for the lateral component of turbulence is given in figure 8. There is complete disagreement with MacPhail's data and Hall's single point, and on the other hand, good agreement with the Dryden-Schubauer single point. Thus there is the curious result that MacPhail's and Hall's data agree well with theory for the longitudinal component and disagree entirely for the lateral component, whereas the converse is true for the Dryden-Schubauer data.

Discussion.—The uncertainty both in the manner of estimating the decay effect and in the data (table I) on which the estimate was based is still far from sufficient to account for the discrepancies between theory and experiment for the lateral component of turbulence. The very large amplification found by MacPhail is particularly hard to explain. On the other hand, the experimental data of the several observers show considerable disagreement, especially when differences in decay are allowed for. This disagreement would tend to cast doubt on the validity of some of the data; the disagreement may also be in part a consequence, predicted by the theory, of possible differences of the initial spectrums from each other and from isotropy.

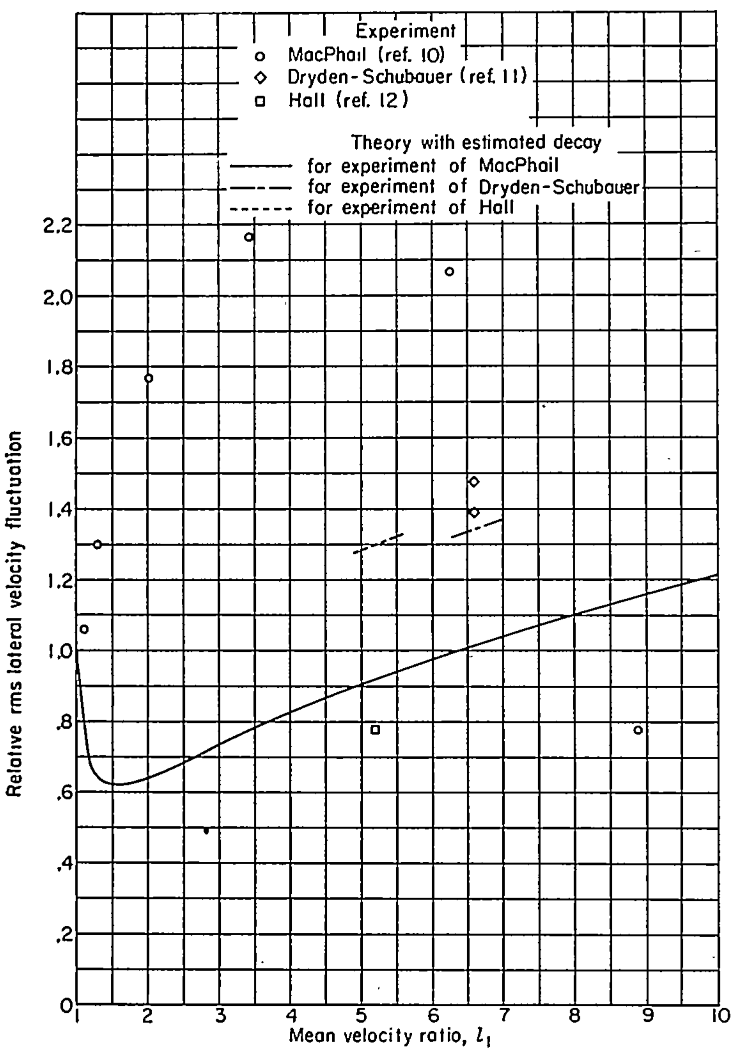


FIGURE 8.—Comparison of predicted axisymmetric contraction effect with experiment for lateral component of turbulence, with decay allowed for in first approximation. Initial isotropic turbulence assumed.

CONCLUDING REMARKS

The original aim of this paper was to provide a quantitative explanation of the observed changes in the root-mean-square velocity components of the turbulence of a wind-tunnel stream after passing through the tunnel contraction. The simplifying assumption of negligible decay was made to make the analysis tractable, although the decay and contraction effects are ordinarily comparable. The analysis on this basis disclosed, in addition to the above integrated effects, pronounced changes in the spectrum of the turbulence. The changes in the shape of the spectral density curves, as distinguished from over-all changes in amplitude, would appear to be considerably less sensitive to modification by decay than would the mean-square velocity components. For this reason, and because such spectral changes have not previously been discussed, the emphasis of the present paper has been placed most heavily on these spectral effects.

In particular, it has been found that the one-dimensional longitudinal spectrum for isotropic turbulence exhibits a rather interesting change in shape downstream of the contraction; the center of gravity of the curve of spectral density versus longitudinal wave number is shifted substantially to higher wave numbers, the resulting distortion moving the peak of the curve well to the right of its initial position above the origin. The distortion is quite pronounced and would appear to be readily amenable to experimental observation.

The restrictive assumption of negligible decay largely defeats the original aim of the paper. Nevertheless, for practical reasons an attempt has been made to provide a crude extension to the theory in which decay is allowed for in first approximation. With this approximation the theory has been compared with experimental values of the contraction effect on the longitudinal and lateral component root-mean-square velocity fluctuations. The agreement for the longitudinal component is good, whereas there appears to be almost complete disagreement for the lateral component, the experimental data themselves being in conflict. It is perhaps premature to attempt any general conclusion. For the present, the theory as augmented by the estimated decay effect may be useful in wind-tunnel-design applications.

It is clear that the tunnel contraction effect on the components of turbulent intensity cannot be represented by fixed fractional changes independent of the character of the initial turbulence. Instead the separate factors for the longitudinal and lateral components depend markedly on the spectrum of the turbulence. For initial isotropy, however, unique factors are predicted that, when decay is neglected, are independent of the details of the spectrum.

LEWIS FLIGHT PROPULSION LABORATORY
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS
CLEVELAND, OHIO, *August 30, 1951*

APPENDIX A

SYMBOLS

The following notation is used in this report:

The subscripts 1, 2, 3 refer to a rectangular coordinate system with the 1-axis alined with the axis of the main flow and directed downstream, the 2-axis directed horizontally, and the 3-axis directed vertically. Separate systems are used with the origins at stations A and B, respectively. (See fig. 1.) Vector and tensor notations are used interchangeably; for example, $\underline{k}=k_\alpha=(k_1, k_2, k_3)$, where $\alpha=1$, 2, or 3 designates a vector with components k_1, k_2, and k_3.

$C(l_1)$	function defined in equation (45)
$D(l_1)$	function defined in equation (44)
D	edge length of cube within which turbulent field is defined
e	base of natural logarithms
$F_\alpha=$	F_1, F_2, or F_3
F_1	one-dimensional longitudinal spectral density (see equation (7))
F_2,F_3	one-dimensional lateral spectral densities (see equation (7))
$G(k)$	function appearing in isotropic spectrum tensor
H	constant $\left(\int_0^\infty k^4G(k)\,dk\right)$
Im	imaginary part of
$i=$	$\sqrt{-1}$
K_1, K_2, K_3	$=k_1+k_1'$, k_2+k_2', k_3+k_3', respectively
k	amplitude of $\underline{k}(=\sqrt{k_1^2+k_2^2+k_3^2})$
$\underline{k}=$	$k_\alpha=(k_1, k_2, k_3)$ wave number vector (station A)
L	lateral scale
l_α	l_1, l_2, l_3
l_1	stream velocity at station B divided by stream velocity at station A (see fig. 1)
l_2	stream breadth at station B divided by stream breadth at station A (see fig. 1)
l_3	stream height at station B divided by stream height at station A (see fig. 1)
M	Mach number of main stream
N	amplitude of special isotropic spectrum tensor (see following equation (26))
$\underline{Q}=$	$Q_\alpha=(Q_1, Q_2, Q_3)$ disturbance wave amplitude vector
$\underline{q}=$	$q_\alpha=(q_1, q_2, q_3)$ disturbance velocity vector
$R_{\alpha\beta}(\underline{r})$	correlation tensor (ref. 6)
Re	real part of
r	magnitude of $\underline{r}=\sqrt{r_1^2+r_2^2+r_3^2}$
$r=$	$r_\alpha=(r_1, r_2, r_3)$ separation vector of two correlated points
s	parameter in equation (28) $\left(=\frac{k_1^2}{\gamma^2}+1\right)$
t	parameter in equation (28) $\left(=\frac{(\epsilon-1)k_1^2}{\gamma^2}-1\right)$
t	time
U	main-stream velocity
$u,v,w=$	q_1, q_2, q_3 disturbance velocity components
X	length of wind-tunnel contraction (distance between stations A and B)
x	used occasionally in place of x_1
$\underline{x}=$	$x_\alpha=(x_1, x_2, x_3)$ position vector (station A)
$\Gamma_{\alpha\beta}(\underline{k})$	spectrum tensor (ref. 3)
γ	constant in special isotropic spectrum tensor (see following equation (26)) ($\gamma=1/L$)
ϵ	contraction parameter ($=l_2^2/l_1^2$; see fig. 1)
$\epsilon_{\alpha\beta\gamma}$	alternating tensor defined after equation (9)
θ	polar angle (equation (31))
κ	magnitude of $\underline{\kappa}(=\sqrt{\kappa_1^2+\kappa_2^2+\kappa_3^2})$
$\underline{\kappa}=$	$\kappa_\alpha=(\kappa_1, \kappa_2, \kappa_3)$ transformed wave number vector (station B); ($\kappa_\alpha=k_\alpha/l_\alpha$)
ν	kinematic viscosity
ξ	magnitude of $\underline{\xi}(=\sqrt{\xi_1^2+\xi_2^2+\xi_3^2})$
$\underline{\xi}=$	$\xi_\alpha=(\xi_1, \xi_2, \xi_3)$ transformed position vector (station B) (see equation (10) and fig. 1(b))
$\sum_\alpha$	summation over α for $\alpha=1$, 2, 3
σ	stream density at station B divided by stream density at station A
τ	a volume
φ	azimuth angle (equation (31))
$\underline{\Omega}=$	$\Omega_\alpha=(\Omega_1, \Omega_2, \Omega_3)$ vector amplitude of vorticity wave
$\underline{\omega}=$	$\omega_\alpha=(\omega_1, \omega_2, \omega_3)$ vorticity vector

Superscripts:

A	measured in vicinity of station A, upstream of contraction
B	measured in vicinity of station B, downstream of contraction
*	complex conjugate

Subscripts:

A	measured in vicinity of station A, upstream of contraction
B	measured in vicinity of station B, downstream of contraction
$\alpha,\beta,\gamma,\delta$	take on values 1, 2, or 3 and designate tensor quantities
1,2,3	specific values of α, β, γ, or δ

A symbol with the mark ~ above it refers to a single plane wave. A bar over a symbol designates an average (usually a spatial average); a bar under a symbol designates a vector.

APPENDIX B

EQUIVALENCE OF SPACE AND TIME AVERAGES IN STATISTICALLY STEADY, HOMOGENEOUS TURBULENCE

The definitions of statistical homogeneity and statistical time independence will first be made precise. Let $F(x, y, z, t)$ be some property of a turbulent field that varies in time and from point to point; thus F may be the pressure, or any of the velocity components, or a correlation of velocity components at two points of fixed séparation, (x, y, z) being one of the two points. If, for all choices of the property F, (a) the average of F over a time $T \to \infty$ is independent of (x, y, z), the turbulence is defined to be statistically homogeneous; if (b) the average of F over a volume $V \to \infty$ is independent of t, the turbulence is defined, in the sense used herein, to be statistically steady or time-independent. The respective averages are supposed to be approached uniformly, in the mathematical sense, as T or V, respectively, approaches infinity. (A statistically steady or "stationary" condition is defined differently in the theory of random processes.)

It will now be proved that if the turbulence satisfies the two conditions (a) and (b), the time and space averages defined therein are equal. In this proof, no resort will be made to the "ergodic hypothesis" of statistical mechanics, which leads to the equivalence of the time average and the "ensemble" average. The possibility of the joint existence of the conditions (a) and (b) probably amounts, however, to just as fundamental an assumption.

The space average will be made over a parallelepiped of edges a, b, c and the time average over a time T, and then a limiting process will be applied. The average of F over both space and time is thus

$$\overline{F}_{s,t}=\lim_{a,b,c,T\to\infty}\frac{1}{abcT}\int_0^T\int_0^c\int_0^b\int_0^a F\,dx\,dy\,dz\,dt \tag{B1}$$

Any order of integration is permissible, since the integration limits are constants. If the time integration is performed first the expression may be written

$$\overline{F}_{s,t}=\lim_{a,b,c,T\to\infty}\frac{1}{abcT}\int_0^c\int_0^b\int_0^a\left(\int_0^T F\,dt\right)dx\,dy\,dz$$

By virtue of the postulated uniform convergence of the time and space averages, the operation $\lim\limits_{T\to\infty}$ may be brought under the integral sign:

$$\begin{aligned}\overline{F}_{s,t}&=\lim_{a,b,c\to\infty}\frac{1}{abc}\int_0^c\int_0^b\int_0^a\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T F\,dt\right)dx\,dy\,dz\\ &=\lim_{a,b,c\to\infty}\frac{1}{abc}\int_0^c\int_0^b\int_0^a\overline{F}_t\,dx\,dy\,dz\end{aligned} \tag{B2}$$

where $\overline{F}_t$ is the time average of F. But, by condition (a), $\overline{F}_t$ is independent of x, y, and z. Therefore

$$\overline{F}_{s,t}=\overline{F}_t \tag{B3}$$

Alternatively, the space integration and limiting process may be performed first:

$$\begin{aligned}\overline{F}_{s,t}&=\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(\lim_{a,b,c\to\infty}\frac{1}{abc}\int_0^c\int_0^b\int_0^a F\,dx\,dy\,dz\right)dt\\ &=\lim_{T\to\infty}\frac{1}{T}\int_0^T\overline{F}_s\,dt\end{aligned} \tag{B4}$$

where $\overline{F}_s$ is the space average of F. By condition (b), $\overline{F}_s$ is independent of t; therefore

$$\overline{F}_{s,t}=\overline{F}_s \tag{B5}$$

Equations (B3) and (B5) together state that

$$\overline{F}_s=\overline{F}_t=\overline{F}_{s,t} \tag{B6}$$

or the *space average*, the *time average*, and the *space-time average* are all equal.

REFERENCES

1. Prandtl, L.: Attaining a Steady Air Stream in Wind Tunnels. NACA TM 726, 1933.
2. Taylor, G. I.: Turbulence in a Contracting Stream. Z. a. M. M., Bd. 15, Heft 1, Feb. 1935, pp. 91–96.
3. Batchelor, G. K.: The Role of Big Eddies in Homogeneous Turbulence. Proc. Roy. Soc. (London), ser. A, vol. 195, Feb. 3, 1949, pp. 513–532.
4. Taylor, G. I., and Batchelor, G. K.: The Effect of Wire Gauze on Small Disturbances in a Uniform Stream. Quart. Jour. Mech. Appl. Math., vol. II, 1949, pp. 1–29.
5. Dryden, Hugh L.: A review of the Statistical Theory of Turbulence. Quart. Appl. Math., vol. 1, 1943, pp. 7–42.
6. de Kármán, Theodore, and Howarth, Leslie: On the Statistical Theory of Isotropic Turbulence. Proc. Roy. Soc. (London), ser. A, vol. 164, Jan. 21, 1938, pp. 192–215.
7. Taylor, G. I.: The Spectrum of Turbulence. Proc. Roy. Soc. (London), ser. A, vol. 164, Feb. 18, 1938, pp. 476–490.
8. Lamb, Horace: Hydrodynamics. Sixth ed., Cambridge Univ. Press, 1932.
9. Dryden, Hugh L., and Abbott, Ira H.: The Design of Low-Turbulence Wind Tunnels. NACA Rep. 940, 1949. (Supersedes NACA TN 1755.)
10. MacPhail, D. C.: Turbulence Changes in Contracting and Distorted Passages. Rep. No. Aero. 1928, British R. A. E., March 1944.
11. Dryden, Hugh L., and Schubauer, G. B.: The Use of Damping Screens for the Reduction of Wind-Tunnel Turbulence. Jour. Aero. Sci., vol. 14, no. 4, April 1947.
12. Hall, A. A.: Measurements of the Intensity and Scale of Turbulence. R. & M. No. 1842, British A. R. C., 1938.

TABLE 1.—DATA FOR ESTIMATION OF DECAY

Source	l_1	U_A, ft/sec	L_A, ft	t, sec	$\sqrt{\overline{u_A^2}}$, ft/sec
MacPhail (ref. 10)	1.20	3.55	0.012	0.19	0.149
	1.60	3.55	.012	.34	-------
	2.55	3.55	.012	.45	-------
	4.90	3.55	.012	.51	-------
	9.65	3.55	.012	.57	-------
Dryden-Schubauer (ref. 11)	6.6	6.86	.05	1.31	.114
Hall (ref. 12)	5.2	1.54	.025	1.22	.046